On the Converse of Hartwick’s Rule: Efficient Constant Utility Path with Zero Net Investment and its Existence

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Abstract

This paper investigates efficient constant utility path in an economy with capital stocks and environmental resources from the viewpoint of Hartwick’s rule. We first show that “maximum constant utility path (MCUP)” is efficient and that it always brings about zero net investment throughout the path if (and only if in the case of non-sigle environmental resources) production is always positive along MCUP. That is, the converse of Hartwick rule holds. Moreover, we demonstrate that MCUP exists. Also, we give the conditions that ensure positive production. This means that, under such conditions, the path satisfying Hartwick’s rule exists.

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1 Introduction

Growing concern about environmental problems sheds light on the analysis of intertemporal resource allocations including environmental resources and the conditions for satisfying intergenerational equity. This is at the core of sustainable development and the pursuit of this issue inevitably pays attention to the well-known Hartwick’s rule as securing sustainable development. Hartwick’s rule, which is discussed by Hartwick (1977) in an economy with exhaustible resources, says that if the economy invests all rents from the use of exhaustible resources in reproducible capital, that is, if “net investment” is zero, a constant consumption path emerges.

This simple but surprising result has given rise to an extensive literature concerning Hartwick’s rule. In particular, Dixit, Hammond, and Hoel (1980) generalize Hartwick’s rule in a competitive economy. On the other hand, the converse of the rule is shown to hold if efficient constant utility path can be characterized as a regular maximin path by Dixit et al., Becker (1982), and by Withagen and Asheim (1997) in a generalized economy under very weak assumptions. That is, they show that if there exists a regular maximin path, then net investment is always zero along the path.

However, it is unknown not only whether the converse of Hartwick’s rule holds even if efficient constant utility path is not a regular maximin path, but also even under what circumstances efficient constant utility path, sat-
isfying Hartwick’s rule, exists. ¹ ² These clarifications are necessary to fully establish a core of sustainable development and also to apply to other arguments relevant to the core, for example, as we remark it in the end of this paper. This paper attempts these clarifications by focusing on the converse of Hartwick’s rule: we first show that under some assumptions, maximum constant utility path (MCUP), that is shown to be efficient, always brings about zero net investment throughout the path if (and only if in the case of non-single environmental resources) production is always positive along MCUP. That is, under the condition, zero net investment emerges along efficient constant utility path. This says that the converse of Hartwick’s rule holds even if MCUP is not characterized as a regular maximin path. This result also implies a possibility that MCUP violates Hartwick’s rule if MCUP has zero production at some time along the path.

Secondly, we demonstrate that MCUP exists if it is feasible to keep consumption more than or equal to some positive amount from the initial time to forever, that is, if the total output from the initial time to forever can be unbounded. This implies, therefore, the existence of a path satisfying Hartwick’s rule under same conditions that make for the converse of Hartwick’s rule. Finally, we show that positive production along MCUP is ensured if all the environmental resources are non-renewable or if the “Inada condition” is satisfied.

¹ Solow(1974) provides us with the existence of a path satisfying Hartwick’s rule in a Cobb-Douglas economy. On the other hand, Dasgupta and Mitra(1983) generalize Solow’s model and show that Hartwick’s rule does not hold in a discrete time economy, although they also show that net investment approaches zero along efficient constant consumption path.

² With respect to a regular maximin path, see Dixit et.al(1980). Dasgupta and Mitra(1983) give a condition to make efficient constant consumption path a regular path.
2 The model

We study an economy with $M (\geq 1)$ forms of capital and $N (\geq 1)$ environmental resources without technological progress and population growth. Let $k_i(t)$ and $S_j(t)$ denote the $i$th capital stock and $j$th environmental resource at time $t$, respectively. Let $(k_1, \cdots, k_M)$ be denoted by $k$, $(S_1, \cdots, S_N)$ by $S$, and $(k, S)$ by $\omega$. We assume that there is an upper bound of $S_j$ for each $j$, which we denote by $\bar{S}_j$. Here, $\dot{S}_j$ is linked with $\omega$ and $z$ as:

$$\dot{S}_j = h_j(\omega) - z_j$$

where $h_j(\omega) (\geq 0)$ is instantaneous natural growth of $j$th environmental resource and $z_j$ is the level of use of the resource in production. We say that an environmental resource is non-renewable if $h_j = 0$ for all $\omega \in R_+^M \times \prod_{j=1}^N [0, \bar{S}_j]$. Otherwise, it is renewable. On the other hand, let $x(t)$ denote a vector on which utility at time $t$ is dependent as $u = u(x(t))$. Here, we assume that $x = (c, S)$ and a strictly concave utility function $u : R_+ \times \prod_{j=1}^N [0, \bar{S}_j] \to R$ satisfies the following property.

$$u_c > 0 \text{ and } u_{S_j} \geq 0.$$  (2)

where $u$ is assumed to be differentiable on $R_+ \times \prod_{j=1}^N (0, \bar{S}_j)$. On the other hand, we assume that this economy has a stationary and convex production possibility set that is represented by $Y$. Recalling (1), we can express that a path $(c(s), \omega(s), \dot{\omega}(s))_{s=0}^\infty$ is feasible if and only if $(c(t), \omega(t), \dot{\omega}(t)) \in Y$ for each $t$. It is convenient below to assume that the set $Y$ can be expressed as:

$$Y = \{ (c, \omega, \dot{\omega}) | f(c, \omega, \dot{\omega}) \leq 0 \}$$  (3)
where \( f \) is assumed to be continuously differentiable with respect to each variable, satisfying that \( f_c > 0 \) and \( f_{k_i} > 0 \). In our economy, if \( z_j = 0 \) for some \( j \), then \( c = -f_c^{-1} \sum_{i=1}^{M} f_{k_i} \dot{k}_i \), that is, there is no production. In other words, every environmental resource is indispensable in production. We assume that if we have no production, there is a \( Q \) such that

\[
f_c^{-1} f_{k_i} \leq Q < \infty \quad (\forall i).
\]

This assumption implies that a path along which \( c(s) \geq \epsilon > 0 \) (\( \forall s \geq 0 \)) is infeasible if production is always zero along the path. Also, it is assumed that \( f_{k_i} < 0 \) and \( f_{S_j} > 0 \) if and only if \( (k-i, z) \gg 0 \) and \( (k, z-j) \gg 0 \).

Moreover, we assume that \( f \) is a strictly convex function. We denote the set of feasible paths starting from \( \omega(t) \) by \( P(\omega(t)) \); that is,

\[
P(\omega(t)) = \{(c(s), \omega(s), \dot{\omega}(s))^\infty_{s=t} | f(c(s), \omega(s), \dot{\omega}(s)) \leq 0 \quad \text{and} \quad (c(s), z(s), \omega(s)) \geq 0 \quad (\forall s \geq t) \}.
\]

Without loss of generality, our analysis throughout the paper is concerned only with feasible allocations satisfying \( f = 0 \).

## 3 MCUP and the converse of Hartwick’s rule

In this section, we investigate the converse of Hartwick’s rule. First of all, we explore the properties of maximum constant utility path as we define it below. We call a path maximum constant utility path (MCUP) if it generates maximum constant utility \( u^* \) among feasible paths given \( \omega(t) \), and denote it by \( P^* = (c^*(s), \omega^*(s), \dot{\omega}^*(s))^\infty_{s=t} \). Note that a constant utility \( u^m \) defined as

\[
u^m = u(0, S(0)),
\]

\[\text{Let } x \text{ be a vector with } n \text{ elements. } x \gg x' \text{ means } x_i > x'_i (\forall i = 1, \ldots, n). \text{ Also, } x_{-i} \text{ represents a vector } (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) (i = 1, \ldots, n) \text{ with } n-1 \text{ elements.}$$
is feasible to achieve. We assume that MCUP yields a constant utility which is higher than \( u^m \). That is,

**Assumption 1** \( u^* > u^m \).

Thus, \( c^*(0) > 0 \). We denote the level of constant utility starting from \( \omega \) by \( V(\omega) \). That is,

\[
V(\omega(t)) = \max_{P \in P(\omega(t))} \bar{u} : u(x(s)) = \bar{u}(\forall s \geq t).
\] (7)

In our economy, \( V \) can be shown to be continuous.

**Lemma 1** \( V(\omega) \) is continuous with respect to \( \omega \).

**Proof:** Let us consider a sequence \( (\omega^\nu)_{\nu=0}^{\infty} \rightarrow \omega \). This sequence accompanies sequences of MCUP at each time \( t \) and the highest attainable constant utility, i.e., \( (\bar{P}^\nu)_{\nu=0}^{\infty} \), and \( (\bar{u}^\nu)_{\nu=0}^{\infty} \). On the other hand, let us denote MCUP and its utility level starting from \( \omega \) by \( \bar{P} \) and \( \bar{u} \). Then, since \( f \) and \( u \) are continuous, we have \( (\bar{P}^\nu)_{\nu=0}^{\infty} \rightarrow \bar{P} \) at each \( t \) so that \( (\bar{u}^\nu)_{\nu=0}^{\infty} \rightarrow \bar{u} \). This implies that

\[
\lim_{\nu \rightarrow \infty} V(\omega^\nu) = V(\omega).
\] (8)

Thus, \( V \) is continuous. ||

It is obvious that, if a path is a MCUP, then it holds that

\[
V(\omega^*(t+h)) \geq V(\omega^*(t))
\] (9)

since if (9) is wrong, \( P^* \) is not feasible. The following lemma suggests that (9) always holds with equality. For the argument, let us now define a subset \( \bar{P}(u', h, \omega(t)) \) as:

\[
\bar{P}(u', h, \omega(t)) = \{(c(s), \omega(s), \dot{\omega}(s))_{s=1}^{\infty} \in P(\omega(t)) | u(x(s)) \geq u'(s : t \leq s \leq t + h)\}.
\] (10)
That is, this is the set of feasible paths whose $(u(x(s)))_{s=t}^{t+h}$ is at least equal to $u'$. Using this notation, the following lemma is provided.

**Lemma 2** Along any $P \in \tilde{P}(u^*, h, \omega^*(t))$, $V(\omega(t+h)) \leq V(\omega^*(t))$ for any $h > 0$.

**Proof:** Suppose that there exists some $P \in \tilde{P}(u^*, h, \omega^*(t))$ for some $h$ along which it holds that $u(x(s)) \geq u^*(\forall s : t \leq s \leq t + h)$ and $V(\omega(t+h)) > V(\omega^*(t))$. Then, since $u$ and $V$ are continuous and since $u_c$ is positive, we can construct another path $P'$ in which, it holds for some $\gamma > 0$,

$$u(x'(s)) = u^* + \gamma(\forall s \geq t).$$

(11)

This contradicts the supposition that that $P^*$ is MCUP starting from $\omega^*(t)$.

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In what follows, it is convenient to assume that we can differentiate $V(\omega)$ with respect to each variable. Let us now differentiate $V(\omega)$ with time.

$$\frac{dV(\omega(t))}{dt} = \sum_{i=1}^{M} V_{k_i}(\omega(t))\dot{k}_i(t) + \sum_{j=1}^{N} V_{S_j}(\omega(t))\dot{S}_j(t).$$

(12)

where $V_{k_i} \equiv \partial V/\partial k_i$ and $V_{S_j} \equiv \partial V/\partial S_j$. The following argument shows that MCUP is only concerned with $dV(\omega^*)/dt = 0$.

From (9) and lemma 2, we obtain next lemma.

**Lemma 3** If a path is a MCUP, then $dV(\omega^*(t))/dt$ is maximized by $\dot{\omega}^*(t)$ subject to $c(t) = c^*(t)$ and it is always equal to zero.

**Proof:** Note that $P^* \in \tilde{P}(u^*, h, \omega^*(t))$ for any $h \geq 0$. Since the previous lemma says that $P^*$ must satisfy

$$V(\omega^*(t+h)) - V(\omega^*(t)) \leq 0(\forall h > 0),$$

(13)
so it must hold \((V(\omega^*(t+h)) - V(\omega^*(t)))/h \leq 0\). Thus, along \(P^\ast\), by taking \(h \to 0\), we obtain
\[
\dot{V}(\omega^*(t)) \leq 0,
\tag{14}
\]
where \(u(x(t)) = u^\ast\). On the other hand, \(dV(\omega^*(t))/dt = 0\) along \(P^\ast\) by (9) and lemma 2. Also, since \(S^\ast\) is fixed at time \(t\), \(u(x(t)) = u^\ast\) is equivalent to that \(c(t) = c^\ast(t)\). Then, our claim has been proved. ||

Before stepping further into the analysis of MCUP, we introduce the following assumption.

**Assumption 2** Along \(P^\ast\), there exists at least one \(P \in P(\omega^*(t))\) along which \(dV(\omega^*(t))/dt \neq 0\) for all \(t \geq 0\).

This assumption only denies that we can take any feasible level of \(c(t)\) and \(z(t)\) without reducing \(V\) or that it is impossible to increase \(V\) even if we take any low level of consumption. In other words, this assumption implies that at least one kind of capital stock or one kind of environmental resources is important in egalitarianism, that is, there exists at least one \(i\) or \(j\) such that \(V_{k_i} > 0\) or \(V_{S_j} > 0\).

Moreover, we assume that \(u\) satisfies the property below.

**Assumption 3** If \(c > 0\), then \(u(c, S) > u(0, \bar{S})\) for any \(S \geq 0\).

This assumption means that consumption is essential in human welfare.

Needless to say, if \(u_{S_j} = 0(\forall j)\), then the assumption is satisfied. Also, this assumption tells us that
\[
(c^*(s), \omega^*(s)) \gg 0(\forall s \geq 0).
\tag{15}
and

\[ \lim_{s \to \infty} c^*(s) > 0. \]  \hspace{1cm} (16)

Otherwise, it must hold that either or both \( c^*(s) = 0 \) for some \( s \) and \( \lim_{s \to \infty} c^* = 0 \). This contradicts assumption 1 and 3 from which we have

\[ u^* = u(x^*(0)) > u(0, \bar{S}) \geq u(0, S^*(s)). \]  \hspace{1cm} (17)

Next lemma is concerned with \( V_{k_i} \).

**Lemma 4** Under assumptions 1 and 2, there is no time \( t \) along \( P^* \) such that it holds that \( V_{k_i}(\omega^*(s)) = 0 \) for any \( i \) at each \( s \geq t \).

**Proof:** Suppose that the claim is not true.

\[
\frac{dV(\omega^*)}{dt} = \sum_{j \in G(s)} V_{S_j}(\omega^*(s))\dot{S}^*_j(s) = 0 (\forall s \geq t)
\]  \hspace{1cm} (18)

where \( G(s) \) is the set of \( js \) that satisfy \( V_{S_j} > 0 \) at time \( s \), the existence of such \( j \) is ensured by assumption 2. If \( h_j = 0 \) for any \( j \in G \), then it must hold that \( \dot{S}^*_j(s) = 0(z^*_j(s) = 0) \) for \( j \in G \) and \( c^*(s) = -f^{-1}_c \sum_{i=1}^M f_{k_i} \dot{k}^*_i \) so that \( \lim_{s \to \infty} c^*(s) = 0 \), by (4). This contradicts (16). On the other hand, if there is some \( j' \in G(s) \) such that \( h_{j'} > 0 \), then it holds either (1) that \( \dot{S}^*_j = 0 (\forall j) \) or (2) that there exist some \( i \) with \( \dot{S}^*_i < 0 \) and some \( j \) with \( \dot{S}^*_j > 0 \). But in both cases, by taking \( d\dot{S}^*_j > 0 \) (in case (1)) or \( d\dot{S}^*_i > 0 \) (in case (2)) in addition to that \( d\dot{k}^*_i < 0 (\forall i) \), we can raise \( \dot{V} \) to positive. This contradicts the fact that \( \dot{V} = 0 \) along MCUP. \( \Box \)

Using this lemma, we explore the efficiency of MCUP.

**Proposition 1** Under assumptions 1, 2 and 3, MCUP is efficient and it is unique.
**Proof:** Recall that (15) and (16) hold along MCUP under assumptions 1 and 3. If $P^*$ is not efficient, then there obviously exists a path $P' \in P(\omega(0))$ that has some interval $T = (T_1, T_2)$

$$u(x'(s)) \geq u^*(\forall s \geq 0)$$

$$u(x'(s)) \geq u^* + \epsilon(s \in T).$$

Then, we can construct another utility path $P^{**} \in P(\omega(0))$ as follows:

$$c^{**}(s) = c'(s) - \Delta(s),$$

$$z_j^{**}(s) = z'_j(\forall s \geq 0),$$

$$u(c^{**}(s), S^{**}(s)) = u^*,$$

$$\dot{k}_i^{**}(s) = \dot{k}_i'(s) + \Delta_i(s),$$

$$\sum_{i=1}^{M} \Delta_i(s) = \Delta(s)$$

where $\Delta_i(s) > 0(\forall i)$ for all $s \in T$ and $\Delta(s) = 0$, otherwise. By assumption 3, it is possible to attain $u(x^{**}(s)) = u^*$ for all $s \geq 0$. By its construction, $P^{**}$ is another MCUP with $\dot{k}_i^{**}(s) > \dot{k}_i'(s)$ for any $s \geq 0$.

On the other hand, since $P'$ ensures that $u'(s) \geq u^*(\forall s \geq 0)$, it must hold for any $s$,

$$V(\omega^{**}(s)) \leq V(\omega'(s)).$$

But notice that $k_i^{**}(s) > k_i'(s) (s > T_1)$ and $S^{**}(s) = S'(s)$ by the construction of $P^{**}$. Thus, it must hold for any $s > T_1$ that

$$V_{k_i}(\omega^{**}(s)) = 0(\forall i).$$

But this contradicts lemma 4 which says that there exist some $t > T_1$ and $i$ such that $V_{k_i} > 0$ along MCUP. This argument shows that MCUP is efficient.
Suppose that MCUP is not unique. Then, any convex combinations of two MCUPs yield paths which dominate the MCUPs in Pareto sense, which contradicts the efficiency of MCUP given by proposition 1.

Now that \( P^* \) is revealed to be efficient, we next show net investment is zero along \( P^* \). From the above proposition, next lemma is derived. We denote a vector of partial derivatives \( (V_{k_1}, \ldots, V_{k_M}, V_{s_1}, \ldots, V_{s_N}) \) at \( \omega \) by \( \Delta V(\omega) \).

**Lemma 5** Under assumptions 1, 2, and 3, \( \Delta V(\omega^*(s)) \gg 0 \) for any \( s \geq 0 \).

**Proof:** Suppose that we have \( V_{k_i'} = 0 \) and/or \( V_{s_j'} = 0 \) for some \( i' \) and \( j' \) at some time along MCUP. Then any revision of \( \dot{k}_{i'}^* \) and \( \dot{s}_{j'}^* \) as far as they are feasible, would attain \( \dot{V} = 0 \). But this contradicts the fact that MCUP exists uniquely so that it is only \( \dot{\omega}^*(s) \) that attains \( dV/dt = 0 \) subject to \( c = c^* \) as indicated by lemma 3.

Using the arguments up to here, we can obtain the following necessary condition for a path to be MCUP.

**Lemma 6** Under assumptions 1, 2 and 3, along MCUP, it holds that (i) \( V_{k_i} = V_{k_j}f_{k_i}/f_{k_j} \) for any \( i, j \) and (ii) \( V_{s_j} \geq V_{k_i}f_{s_j}/f_{k_i} \) for any \( i, j \). Strict inequality in (ii) holds only if \( z_j = 0 \).

**Proof:** From lemma 3, \( \dot{\omega}^*(t) \) is the solution of

\[
\max_{\dot{\omega}(t)} \left( \sum_{i=1}^{M} V_{k_i}(\omega^*(t)) \dot{k}_i(t) + \sum_{j=1}^{N} V_{s_j}(\omega^*(t)) \dot{s}_j(t) \right)
\]

subject to

\[
f(c(t), \omega^*(t), \dot{\omega}(t)) = 0 \text{ and } c(t) = c^*(t).
\]

To solve this problem easily leads us to, since \( \omega^* \gg 0 \) by (15),

\[
V_{k_j}(\omega^*(t)) f_{k_i} - V_{k_i}(\omega^*(t)) f_{k_j} = 0.
\]
Moreover,

\[ V_k(\omega^*(t)) f_{S_j} - V_{S_j}(\omega^*(t)) f_{k_i} \leq 0. \]  

(26)

It is obvious that (26) holds with strict inequality only if \( z_j = 0 \), since \( \Delta V(\omega^*(s)) \gg 0 \) by lemma 5 and \( f_{k_i} > 0 (\forall i) \). ||

\( V_{S_j}/V_{k_i} \) expresses the marginal rate of substitution between \( S_j \) and \( k_i \) along \( P^* \). Hence, (ii) in the above lemma says that, if it holds with equality, the rate is equivalent to the competitive price of environmental resource \( S_j \) measured by that of capital stock \( k_i \).  

Now we can show that the relationship of maximum constant utility path with “net investment” \( I(t) \) at time \( t \) as defined below.

\[ I(t) = (f_{k_1})^{-1} \left( \sum_{i=1}^{M} f_{k_i}(t) \dot{k}_i(t) + \sum_{j=1}^{N} f_{S_j}(t) \dot{S}_j(t) \right). \]  

(27)

\( I \) is interpreted as (competitive) market value of net depreciation of endowment of resources measured by the price of 1st capital stock. Zero net investment, therefore, implies that the value of resources lost at that time is nil.

In what follows, we say that production is positive if \( (k, z) \gg 0 \). Moreover, if a constant utility path is efficient and yields zero net investment along the path, the path is referred to as “Hartwick path”.

**Theorem 1** Under assumptions 1, 2 and 3, MCUP is equivalent to Hartwick path if production is positive at each time along the path. Moreover, if \( N \geq 2 \), the equivalence holds only if MCUP has positive production at each time.

**Proof:** Along MCUP, lemma 3 says that it must hold for any \( s \geq 0 \) that

\[ dV(\omega^*(s))/ds = 0. \]  

(28)

\footnote{This property is also provided in Becker(1982) in a pollution economy.}
On the other hand, from lemma 6, (ii) in lemma 6 holds for all $j$ with equality if $(k^*(s), z^*(s)) \gg 0$ along the path. So, if this is the case, by substituting (i) and (ii) in lemma 6 into (28), we obtain for any $s$ that

$$dV(\omega^*(s))/dt = V_{k_1} I^*(s).$$

Since $V_{k_1}(\omega^*(t)) > 0$ from lemma 5, we have

$$I^*(t) = 0(\forall t \geq 0).$$

On the other hand, suppose that it holds that $V_{S_j} = V_{k_1} f_{S_j}/f_{k_1}$ at $z^*_j = 0$ in the case that $N \geq 2$. But this means that $z^*_{-j} \gg 0$ because $f_{S_j} > 0$ must hold. If so, however, we can raise $\dot{S}_i$, so $\dot{V}$ as well. This contradicts the fact that $z^*$ maximizes $\dot{V}$ as indicated by lemma 3. Thus, (ii) in lemma holds only if $z^* \gg 0$. ||

This proposition is exactly the converse of Hartwick’s rule; efficient constant utility path yields zero net investment along the path. On the other hand, for example, if $N \geq 2$ and production is not positive at some time along MCUP, then we have (ii) in lemma 6 with strict inequality so that we have at that time

$$\dot{V} = 0 > V_{k_1} I^*.$$ 

That is, net investment is negative. The proposition shows that positive production is crucial for the converse to hold. We discuss of sufficient conditions that ensures positive production later.

4 Existence of “Hartwick path”

In this section, we explore the existence of Hartwick path. Since Hartwick path is efficient and since constant utility path bringing about
\( u^m = u(0, S(0)) \) is inefficient, the path exists only if \( u^* > u^m \). This condition is supposed by assumption 1, but we now introduce the following assumption and derive \( u^* > u^m \).

**Assumption 4** There is a path \( P \in P(\omega(0)) \) such that \( \inf_{t \geq 0} c(t) \geq \epsilon > 0 \) for some \( \epsilon > 0 \).

This assumption ensures that total output from time zero to infinity can be unbounded. This condition is in particular essential in the analysis of an economy with exhaustible resources and many researches implicitly assume this condition. For Cobb-Douglas production function cases where \( y = k^\alpha z^\beta \) with \( \alpha + \beta < 1 \), it is well-known that this is possible only if \( \alpha > \beta \). \(^5\) Now we define a set \( M \) as:

\[
M = \{ \bar{u} \in \mathbb{R} | u(x(t)) = \bar{u} (\forall t \geq 0) \text{ is feasible} \}.
\] (32)

That is, \( M \) is the set of feasible constant utility levels. Assumption 4 ensures that \( M \setminus \{u^m\} \neq \emptyset \). \(^6\) In fact, if there exists some \( P \) along which \( c(t) \geq \epsilon > 0 (\forall t \geq 0) \), then we can construct a constant utility path \( P^{**} \) that brings about utility \( u_\epsilon \) where

\[
u_\epsilon = \inf_{t \geq 0} u(x(t)), \] (33)

in the same way as in (21), i.e.,

\[
c^{**}(s) = c(s) - \Delta(s), \] (34)

\[
 z_j^{**}(s) = z_j(s)(\forall s \geq 0),
\]

\(^5\)For the detail, see Dasgupta and Heal(1979), ch.7.

\(^6\)This property might be called the existence of “non-trivial” constant utility path following, for example, Dasgupta and Mitra(1983). Under this condition, they show that efficient constant consumption path exists in an economy with exhaustible resources in a discrete time economy. On the other hand, Cass and Mitra(1991) explore the conditions to establish assumption 4.
\begin{align*}
\dot{k}_i^{ss}(s) &= \dot{k}_i(s) + \Delta_i(s), \\
\sum_{i=1}^{M} \Delta_i(s) &= \Delta(s)
\end{align*}

where \( \Delta(s) \) satisfies \( u(c(s) - \Delta(s), S(s)) = u_\epsilon \). Thus,

\[ M - \{u^m\} \neq \emptyset. \quad (35) \]

Now we can show the existence of efficient constant utility path. In an economy with a capital stock and an environmental resource, the existence of efficient constant consumption path is shown by Dasgupta and Mitra(1983) in a discrete time model.

**Proposition 2** Under assumptions 3 and 4, MCUP with \( u^* > u^m \) exists and it is efficient. Along the path, \( c(s) \geq \delta > 0 \) for some \( \delta > 0 \) and \( \omega(s) \gg 0 \) for each \( s \geq 0 \).

**Proof.** From (35), we know that \( M - \{u^m\} \neq \emptyset \). Moreover, \( M \) is bounded from above, since there is large enough \( \bar{c}(0) \) such that \( \bar{u}' = u(\bar{c}(0), S(0)) \notin M \).

Finally, consider a sequence \( (\bar{u}^\nu)_{\nu=0}^{\infty} \rightarrow \bar{u} \) where \( \bar{u}^\nu \in M(\forall \nu \geq 0) \). Then, there exists \( P^\nu \in P(\omega(0)) \) with constant utility \( \bar{u}^\nu \) for each \( \nu \). So, it holds that, for each \( \nu \) and each \( t \),

\[ u(x^\nu(t)) = \bar{u}^\nu \quad (36) \]

\[ f(c^\nu(t), \omega^\nu(t), \dot{\omega}^\nu(t)) = 0. \]

Therefore, by the continuity of \( u \) and \( f \), it must be that \( \bar{u} \in M \) so that \( M \) is closed. Thus, \( M \) has its maximum, which means the existence of MCUP.

Finally, it must hold that

\[ u^* \geq u_\epsilon, \quad (37) \]
where \( u_\epsilon \) is given in (33) so that \( c^*(t) > 0 \) at each \( t \) by assumption 3. Moreover, it is not true that \( \lim_{s \to \infty} c^*(s) = 0 \) by the same argument which derives (16). Finally, it is immediate from this that \( \omega^*(t) \gg 0 \) at each \( t \).

Efficiency of MCUP is obvious from proposition 1, whose suppositions are revealed to be satisfied.

Note that assumption 4 is satisfied when all the environmental resources are renewable; in this case, by taking \( z_j = h_j \), it is indeed feasible to keep \( \omega(t) \gg 0 (\forall t > 0) \) along which \( c(t) \) is a positive constant. Notice also that we do not need assumption 3 for the existence of efficient constant utility path if \( u_{S_j} = 0 (\forall j) \) as well as in the previous section; the assumption is required in this section only for that we can lower utility to attain the level \( u_\epsilon \) by reducing consumption.

Assuming that \( (\omega^*(s))_{s=0}^{\infty} \) is differentiable with respect to time, we already know that MCUP is Hartwick path if production is positive throughout the path. This condition is satisfied if all the environmental resources are non-renewable.

**Theorem 2** Under assumptions 2, 3 and 4, there exists Hartwick path if all the environmental resources are non-renewable.

**Proof:** Since \( \triangle V(\omega^*(s)) \gg 0 \) by lemma 5 and \( \dot{V} = 0 \) along \( P^* \), the existence of some \( j \) such that \( z_j^* = 0 \) implies that,

\[
\sum_{j=1}^{N} V_{S_j} \dot{S}_j^* > 0,
\]

because it holds that, from (i) of lemma 6,

\[
\sum_{i=1}^{M} V_{k_i} \dot{k}_i^* = \sum_{i=1}^{M} V_{k_i} f_{k_i}^{-1}(f_{k_i} \dot{k}_i^*(s)) = V_{k_i} f_{k_i}^{-1}(-f_c c^*(s)) < 0.
\]
The last equation comes from the supposition that production is zero at this time. But since \( h_j = 0 \) for each \( j \), \( \dot{S}_j \leq 0 \), which contradicts (38). So, \( z^* \gg 0 \). Thus, the claim is straightforward from theorem 1 and proposition 2.

Finally, we adopt the following assumption to ensure positive production along MCUP even if some environmental resources are renewable.

**Assumption 5** \( \lim_{z_j \to 0} f_{\dot{S}_j} = \infty \) if \( (k, z_{-j}) \gg 0 \).

This condition may be identified with the “Inada condition” that is often used in neoclassical growth theory. With this assumption, we show next theorem.

**Theorem 3** Under assumptions 2, 3, 4 and 5, there exists Hartwick path.

**Proof:** Suppose that \( N = 1 \) and \( z^* = 0 \) at some time. But if so, by assumption 5, it must hold that \( V_{S_j} < V_{k_i} f_{\dot{S}_j} / f_{k_i} \) so that \( z^* > 0 \).

On the other hand, suppose that \( N \geq 2 \) and there is no production at some time. Since \( V_{S_j} > 0 \) for all \( j \), it must be \( z^* = 0 \) in order \( \dot{\omega}^* \) to maximize \( \dot{V} \) at that time. Let us now take another \( z' \) where \( z'_i = \epsilon (\forall i \neq j) \) and \( z'_j = 0 \).

Then, by our supposition, this allocation leads to

\[
\dot{V} = \sum_{i=1}^{M} V_{k_i} \dot{k}_i^* + V_{S_j} \dot{S}_j^* + \sum_{i \neq j} V_{S_i} (\dot{S}_i^* - \epsilon) \\
= 0 - \sum_{i \neq j} V_{S_i} \epsilon.
\]

But since \( f_{\dot{S}_j} = \infty \) at \( z' \) by assumption 5, it holds that at \( z' \)

\[
V_{k_i} f_{\dot{S}_j} / f_{k_i} - V_{S_j} = \infty
\]
so, \( dz'_j > 0 \), it holds that for some \( i \),

\[
(V_k f_{S_j}/f_{k_i} - V_{S_j})dz'_j - \sum_{i \neq j} V_{S_i} \epsilon > 0.
\]

(42)

This contradicts that \( z^* = 0 \) maximizes \( \dot{V} \). Thus, there must be positive production, so that \( z^* \gg 0 \). This immediately means, together with theorem 1 and proposition 2, that Hartwick path exists.

5 Concluding remarks

This paper explores the properties of maximum constant utility path (MCUP) which is shown to be efficient under our assumptions, from the viewpoint of Hartwick’s rule. We find that, as far as MCUP is equipped with the property that production is positive at each time, MCUP yields zero net investment throughout the path. That is, the converse of Hartwick’s rule holds along MCUP. On the other hand, we show that MCUP exists and find the circumstances in which positive production is ensured. Thus, in such circumstances, Hartwick path does exist.

Our main results depend upon assumption 2 which implies that at least one feasible path exists in \( P(\omega^*(t)) \), along which \( dV(\omega^*(t))/dt \neq 0 \). We have not proved this, but if we take a large enough \( \bar{u} \), there exists some \( \epsilon \) satisfying that \( \exists i : k_i(t + \epsilon) = 0 \) along all \( P \in \bar{P}(\bar{u}, \epsilon, \omega^*(t)) \). This implies that \( V(\omega(t + h)) \) in this case is smaller enough than \( u^*(= V(\omega^*(t))) \), so it seems that the assumption poses little problem of plausibility even if we assume \( V_{k_i} > 0 \) instead of assumption 2.

We do not investigate in this paper whether efficient constant utility path can be characterized as a regular path. Dasgupta and Mitra (1983)’s analysis
implies that we may need other conditions on production possibility function to prove this. If the characterization holds, Withagen and Asheim(1997)’s analysis is more general than ours regarding the converse of Hartwick’s rule.

Finally, the result that efficient constant utility path exists in a continuous time model as well as in a discrete time model, may be some basis for other discussions. For example, let us suppose that we compare two resources $\omega_1$ and $\omega_2$ and judge which is better based upon the levels of $V(\omega_1)$ and $V(\omega_2)$. Suppose also that we maximize $\int_0^\infty u(x(s))e^{-rs}ds$ given $\omega(0)$ and denote the optimal path by $P^u$. Then, how does $P^u$ should be characterized in order for it to have such a property that $V(\omega^u(s)) \geq V(\omega^u(t))$ for all $s \geq t (\forall t \geq 0)$ along $P^u$, that is, $dV(\omega^u(t))/dt \geq 0$ for all $t$? Using the efficiency property of MCUP, it is easy to show that it must hold that $u(x^u(s)) \leq V(\omega^u(s))$ with some interval during which strict inequality holds. Further analysis from this viewpoint might need the property that efficient constant utility path yields zero net investment.

References


Dasgupta, P. and G. M. Heal(1979), Economic theory and exhaustible re-

\footnote{The analysis along this line is partly discussed in Onuma(1998).}
sources, Cambridge University Press.


