Technological Change and Market Dynamics

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Preliminary Version

May 29, 1998

Abstract

Nonrenewable resource scarcity has been a traditional concern when designing optimal growth models. Technological change has played an important role in those models, since its presence mitigates the depletion effect on extraction paths over time. Moreover, in empirical research, when studying the time trends of prices and other scarcity measures, the role played by technology is crucial. The effects of technological change have been studied in previous literature in several different ways. More emphasis has been placed on capital-resource growth models than on specific sectoral behaviour models. However, in order to predict actual market dynamics the latter are more appropriate.

In this paper, we will examine the impact of different kinds of technological change on extraction path over time, in the context of a competitive resource industry. Two cases are considered. In the first case, prices are assumed to be exogenous, while in the second demand is introduced. Therefore, both price and extraction paths result from demand/supply dynamics. Implications for the evolution of scarcity measures (price, marginal extraction cost and user cost) are derived.

1. Introduction

Simple models of nonrenewable resource extraction consider the case of a firm that has a fixed production process, implying that the firm’s cost function does

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not change throughout the entire period of extraction activities. However, the assumption of no technical improvements in production is empirically inappropriate for most resources. Technology can be (and historically appears to have been) an important way to decrease the scarcity problems inherent of nonrenewable resources, since it allows the extraction of resource stocks that were previously too expensive. Thus, the analysis of extraction behavior in the presence of technical change is quite pertinent.

The impact of technological change on resource extraction can be modeled in several different ways. When using a basic profit maximization model to analyze resource extraction, technological progress is generally represented by a downward shift in the extraction cost function and/or in the exploration cost function. A number of authors have done that in a simple way by including time as an argument of cost\(^1\). However, this implies that progress is basically an exogenous process, whereas most of the literature suggests that technological change is at least partly endogenous\(^2\). Furthermore, some general results can be obtained even without this simplification.

Farzin [6] tries to study the impact of technological change on the behavior of scarcity measures (price, marginal cost of extraction and scarcity rent) for a competitive firm. The author begins by considering the case of no technological change, and presents the dynamics of scarcity measures (with special emphasis to the relationships between their movements). The author then attempts to extend his results to the case of technological change, but his conclusions rely on using a given extraction path to compare the case of progress with the case of no progress. This seems quite illogical, for it implies that the competitive firm does not change the chosen path for its decision variable even though its cost function has been altered, and hence it is not behaving as a profit maximizer.

In this paper, a more adequate study of technological change is presented, using comparative dynamic methods from Caputo [2] and Sweeney [11]. The basic behavior of scarcity measures is sketched, then the impact of technical progress on those measures (as well as on the extraction path) is calculated. Finally, an attempt is made to take demand into consideration, thus gaining some insight

\(^1\)See, for example, Dagupta and Heal [3], Slade [10], and Berck and Roberts [1]. An example of a specific cost function that is estimated including time is given in Lasserre and Ouellette [9]. The same strategy is often applied to production functions in capital-resource models with consumption, although these have limited interest for empirical studies (again, see Dasgupta and Heal [3]).

\(^2\)The methods used to explain technical progress are similar to those used in investment theory. For a good introduction to the subject, see Kamien and Schwartz [8].
2. A Basic Model of Extraction

Consider the general profit maximization problem for an individual owner of a nonrenewable resource deposit when price is taken as given:

$$\max_{E_t} \Pi = \int_0^T \left[ P_t E_t - C_t(E_t, S_t, Z_t) \right] e^{-rt} dt$$

s.t. \( \dot{S} = -E_t \)

\( S_0 = \Sigma \)

\( S_T \geq 0 \)

\( E_t \geq 0 \)

where \( S_t \) is the existing stock of resource in moment \( t \), \( E_t \) is the extraction at \( t \) and \( Z_t \) indicates the level of technology attained at \( t \) (assumed to be exogenous). \( \Sigma \) is the known initial number of resource units available\(^3\).

The cost function is assumed to have the following properties (all derivatives evaluated at \( t \)):

1. it is continuously differentiable;

2. it is strictly convex, ie. \( \frac{\partial^2 C}{\partial E^2} > 0, \frac{\partial^2 C}{\partial S^2} > 0, \frac{\partial^2 C}{\partial Z^2} > 0 \) and \( \frac{\partial^2 C}{\partial E^2} \frac{\partial^2 C}{\partial S^2} - \left( \frac{\partial^2 C}{\partial E \partial S} \right)^2 > 0 \);

3. according to intuition, in the absence of learning by doing one expects \( \frac{\partial C}{\partial E} > 0, \frac{\partial C}{\partial S} < 0, \frac{\partial C}{\partial Z} < 0, \frac{\partial^2 C}{\partial E \partial S} < 0, \frac{\partial^2 C}{\partial E \partial Z} < 0, \frac{\partial^2 C}{\partial S \partial Z} > 0 \);

\(^3\)Farzin (\[5\] and \[6\]) criticizes the use of a finite, known stock of the resource, and replaces the state equation in 2.1 with \( \dot{X} = E_t \), where \( X_0 = 0 \). However, the necessary assumption that \( \lim_{t \to \infty} X_t = \bar{X} \) seems to mean exactly the same thing if one is trying to calculate the extraction path. Moreover, the author’s justification for the existence of a finite maximum for cumulative extraction is that barring technological change, increasingly large quantities of the resource can be exploited only at increasingly high incremental costs...it will never make economic sense to extract infinite amount of the resource (\[5\], p.815-6). If it is true that the finite initial stock assumption is unrealistic, then a better way to represent the state equation would be to include discoveries, which could encompass the effect of new geological information as well as technological progress.
Thus, marginal extraction cost is positive and increasing (reflecting diminishing returns to extraction); there are stock effects in both total and marginal cost (when the stock decreases, extraction becomes more costly); as for technology, it is assumed to lower total and marginal extraction cost, and to decrease the impact of stock effects on total cost (note that $\frac{\partial C}{\partial S}$ becomes smaller in absolute value when the level of technology improves)

The Hamiltonian (in current value) for this problem is:

$$H_c = P_t E_t - C(E_t, S_t, Z_t) + \phi_t(-E_t)$$

with first order conditions$^5$:

1. $\frac{\partial H_c}{\partial E_t} = \left[ P_t - \frac{\partial C_t}{\partial E_t} \right] - \phi_t \leq 0; \ E_t \geq 0; \ E_t \left( \left[ P_t - \frac{\partial C_t}{\partial E_t} \right] - \phi_t \right) = 0$

2. $\phi_t - \phi_t \tau = -\frac{\partial H_c}{\partial S_t} = \frac{\partial C_t}{\partial S_t}$

3. $\dot{S} = -E_t$

4. $\lim_{t \to T} \phi_t e^{-rt} S_t = 0; \ \lim_{t \to T} \phi_t \geq 0; \ \lim_{t \to T} S_t \geq 0$

Given the properties of the cost function, $H_c$ is concave in $(S_t, E_t)$. Hence these conditions are both necessary and sufficient.

From condition $ii$), solving the differential equation:

$$\phi_t = e^{-r(T-t)} \phi_T - \int_t^T \frac{\partial C}{\partial S} e^{-r(\tau-t)} d\tau$$

Substituting 2.3 into condition $ii$ again, the behaviour of $\phi$ can be summarized through $^6$:

$$\dot{\phi} = e^{-r(T-t)} \left( \phi_T r + \frac{\partial C}{\partial S} \right) - \int_t^T \frac{\partial C}{\partial S} e^{-r(\tau-t)} d\tau$$

When the time horizon is far enough into the future $(T \to \infty)$, 2.4 becomes:

$$\dot{\phi} = -\int_t^\infty \frac{\partial C}{\partial S} e^{-r(\tau-t)} d\tau$$

$^4$These assumptions are similar those found in the literature. See, for example, Farzin [5].

$^5$If the problem has an infinite horizon, then the transversality condition $\lim_{t \to \infty} e^{-rt} H_c = 0$ must also hold, but the discount factor ensures that the condition is verified as long as $H_c$ is finite.

$^6$The expression $\frac{\partial C}{\partial S}$ refers to $\frac{\partial (\frac{\partial C}{\partial S})}{\partial t}$. 

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Equation 2.5 shows the close relationship between the behaviour of stock effects throughout time and that of opportunity cost (namely, if $\frac{\partial \hat{C}}{\partial S} > 0$ for $\tau > t$ then $\dot{\phi} < 0$ at $t$ and vice-versa for $\frac{\partial \hat{C}}{\partial S} < 0$).

Note, from condition $i$), that whenever extraction is positive:

$$P_t = \frac{\partial C_t}{\partial E_t} + \phi_t \quad (2.6)$$

The dynamic version of 2.6 is:

$$\dot{P} = \frac{\partial \dot{C}}{\partial E} + \dot{\phi} \quad (2.7)$$

Note also that:

$$\frac{\partial \hat{C}}{\partial S} = \frac{\partial^2 C}{\partial E \partial S} \hat{E} + \frac{\partial^2 C}{\partial S^2} \hat{S} + \frac{\partial^2 C}{\partial S \partial Z} \hat{Z} \quad (2.8)$$

$$\frac{\partial \hat{C}}{\partial E} = \frac{\partial^2 C}{\partial E^2} \hat{E} + \frac{\partial^2 C}{\partial E \partial S} \hat{S} + \frac{\partial^2 C}{\partial E \partial Z} \hat{Z} \quad (2.9)$$

Equations 2.7, 2.8, 2.9, and 2.5 or 2.4 can be used to study the relationship between opportunity cost, price, and extraction cost along the optimal extraction trajectory.

2.1. Extraction with Parameter Invariance

In this section it will be assumed that the parameters of problem 2.1 (price, interest rate, and the level of technology) remain constant and that the time horizon is arbitrarily large. Under those conditions, equations 2.7, 2.9, and 2.8 can be combined to yield:

$$0 = \frac{\partial \dot{C}}{\partial E} + \dot{\phi} \quad (2.10)$$

$$\frac{\partial \dot{C}}{\partial S} = \frac{\partial^2 C}{\partial E \partial S} \left( \frac{\partial \dot{C}}{\partial E} \right) + \frac{\partial^2 C}{\partial E \partial S} \left[ \left( \frac{\partial^2 C}{\partial E \partial S} \right)^2 - \frac{\partial^2 C}{\partial E^2} \frac{\partial^2 C}{\partial S^2} \right] \quad (2.11)$$

From 2.10, marginal cost and opportunity cost will necessarily be moving in opposite directions along the optimal path. However, with cost function convexity 2.11 shows that if $\frac{\partial C}{\partial E} > 0$ then $\frac{\partial C}{\partial S} < 0$; if that happens for a long enough period of time, $\dot{\phi} > 0$. Hence, for the case of invariant price it will generally be true that
\frac{\partial \dot{C}}{\partial E} < 0 \text{ and } \dot{\phi} > 0, \text{ since } \dot{\phi} < 0 \text{ would mean } \frac{\partial \dot{C}}{\partial S} > 0 \text{ for at least some periods, implying } \frac{\partial \dot{C}}{\partial E} < 0 \text{ which contradicts 2.107. Under these conditions, equation 2.9 indicates that extraction will be decreasing along the optimal trajectory.}

3. Technological Progress with exogenous prices

3.1. Single dZ

In this section the role of technology will be examined in a comparative dynamics perspective, although to begin with the comparative static effects on the steady state shall be ascertained. When the parameters of the model are invariant (price $P$, interest rate $r$, and technological progress $Z$), the system described by the first order conditions will eventually reach a steady state, where $\dot{S} = 0$ and $\dot{\phi} = 0$. This steady state may be characterized either by the exhaustion of the initial resource stock or by the abandonment of remaining units (if price falls below marginal extraction cost plus opportunity cost, extraction becomes unprofitable and $E = 0$, as can be seen from first order condition i)). Note that if the terminal time is sufficiently far into the future it is possible to have both $\bar{S} > 0$ and $\bar{\phi} > 0$ (steady state values) without breaking the transversality condition. This is the case that will be analysed throughout the rest of the paper. It can be shown that under concavity of the Hamiltonian and with $\frac{\partial^2 C}{\partial E \partial S} < 0$ the steady state is a saddle point.

The first order conditions imply that at the steady state:

\[ \bar{\phi} = -\frac{\partial C}{\partial S} \bigg|_{E=0,S=\bar{S}} \]

\[ P = \frac{\partial C}{\partial E} \bigg|_{E=0,S=\bar{S}} + \bar{\phi} \]  \hspace{1cm} (3.1) \hspace{1cm} (3.2)

Now, assuming that the level of technology was higher than in the previous case ($Z' > Z$, hence $dZ > 0$) the effects on the optimal extraction path are examined,

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\footnote{It is possible to have $\frac{\partial \dot{C}}{\partial E} > 0$ and $\dot{\phi} < 0$ for a period of time if $\frac{\partial \dot{C}}{\partial S}$ behaves non-monotonically, namely if it starts out negative (which it will have to remain as long as marginal cost is growing) and then turns positive. Thus it would be possible, but not necessary, for extraction to grow for some time and then start decreasing.}

\footnote{As Sweeney [11] remarks, the situation where $\bar{\phi} > 0$ is usually associated with a negative, stock-dependent externality that would linger even after extraction had ceased.}
beginning with the steady state. A phase diagrammatic approach similar to the one in Sweeney [11] is presented in the Appendix.

Total differentiation of 3.1 and 3.2 yields:

\[ d\bar{\phi} = -\frac{1}{r} \left( \frac{\partial^2 C}{\partial S^2} dS + \frac{\partial^2 C}{\partial S \partial Z} dZ \right) \]  
\[ (3.3) \]

\[ dP = \frac{\partial^2 C}{\partial E \partial Z} dZ + \frac{\partial^2 C}{\partial E \partial S} dS + d\bar{\phi} \]  
\[ (3.4) \]

Maintaining the assumption of constant prices, \( dP = 0 \), equation 3.4 becomes:

\[ d\bar{\phi} = -\frac{\partial^2 C}{\partial E \partial Z} dZ - \frac{\partial^2 C}{\partial E \partial S} dS \]  
\[ (3.5) \]

Solving 3.3 and 3.5, and simplifying yields:

\[ d\bar{\phi} = -\frac{\partial^2 C}{\partial E \partial Z} dZ - \frac{\partial^2 C}{\partial E \partial S} dS \]  
\[ d\bar{\phi} = -\frac{\partial^2 C}{\partial E \partial Z} dZ - \frac{\partial^2 C}{\partial E \partial S} dS \]  
\[ (3.6) \]

\[ dS = \frac{\partial^2 C}{\partial S^2} \frac{\partial^2 C}{\partial S \partial Z} dZ \]  
\[ (3.7) \]

Using the properties assumed for the cost function, it is shown that the steady state stock will decrease with a single technological improvement, which means that cumulative extraction is higher. This result can be explained intuitively if one remembers that extraction costs are lower, which combined with a constant price means that some units that were previously abandoned now become sufficiently attractive to be extracted. In general, nothing can be said about the sign of \( d\bar{\phi} \) when \( dZ > 0 \). The explanation for this indeterminacy lies with the double effect of technology on costs: on one hand, marginal extraction costs are decreased, making the value of each remaining stock unit higher; but on the other hand, the negative impact of depletion on costs becomes smaller, meaning that each unit of remaining stock has less cost-reducing value. A clear illustration of these effects is obtained if one looks at each of them separately, as follows.

Technological progress might not affect both marginal extraction cost and marginal depletion cost. Two special cases will be analyzed\(^9\):

\(^9\)Farzin [6] also presents the case of neutral technological change, where \( \frac{\partial^2 C}{\partial E \partial Z} \) does not change with the level of technology. This case brings no additional information to our model, hence it will not be considered here.
• Extraction-biased technological change: $\frac{\partial^2 C}{\partial E \partial Z} < 0$ and $\frac{\partial^2 C}{\partial S \partial Z} = 0$

In this case equations 3.6 and 3.7 become:

$$d\bar{S} = \frac{-r \frac{\partial^2 C}{\partial E \partial Z}}{r \frac{\partial^2 C}{\partial E \partial S} - \frac{\partial^2 C}{\partial S^2}} dZ$$ \hspace{1cm} (3.8)

$$d\bar{\phi} = \frac{\frac{\partial^2 C}{\partial E \partial Z} \frac{\partial^2 C}{\partial E \partial S} - \frac{\partial^2 C}{\partial S^2}}{r \frac{\partial^2 C}{\partial E \partial S} - \frac{\partial^2 C}{\partial S^2}} dZ$$ \hspace{1cm} (3.9)

Note that the impact on steady state scarcity rent is no longer ambiguous. In the presence of this type of technological change, $\bar{\phi}$ increases with $dZ > 0$.

• Depletion-biased technological change: $\frac{\partial^2 C}{\partial E \partial Z} = 0$ and $\frac{\partial^2 C}{\partial S \partial Z} > 0$

Now the relevant equations are:

$$d\tilde{S} = \frac{\frac{\partial^2 C}{\partial S \partial Z}}{r \frac{\partial^2 C}{\partial E \partial S} - \frac{\partial^2 C}{\partial S^2}} dZ$$ \hspace{1cm} (3.10)

$$d\tilde{\phi} = -\frac{\frac{\partial^2 C}{\partial S \partial Z} \frac{\partial^2 C}{\partial E \partial S} - \frac{\partial^2 C}{\partial S^2}}{r \frac{\partial^2 C}{\partial E \partial S} - \frac{\partial^2 C}{\partial S^2}} dZ$$ \hspace{1cm} (3.11)

Thus, this type of technological change causes a decrease of steady state scarcity rent.

Unlike what has been done so far, a true comparative dynamic analysis requires taking into account impacts on the entire optimal path. For this purpose a variational differential equation approach similar to that in Caputo [2] is appropriate\textsuperscript{10}. It was shown above that 3rd order conditions for the single competitive producer problem, with parameter invariance, are given by:

$$P = \frac{\partial C}{\partial E}(E, S; Z) + \phi$$ \hspace{1cm} (3.12)

$$\dot{\phi} = r\phi + \frac{\partial C}{\partial S}(E, S; Z)$$ \hspace{1cm} (3.13)

$$\dot{S} = -E$$ \hspace{1cm} (3.14)

Condition 3.12 defines optimal extraction:

$$E^* = E(S, \phi; P, Z)$$ \hspace{1cm} (3.15)

\textsuperscript{10}See also Epstein [4] for the infinite horizon case.
The partial derivatives of 3.15 can be calculated implicitly:

\[
\begin{align*}
\frac{\partial E}{\partial S} &= -\frac{\partial^2 C}{\partial E^2} > 0 \\
\frac{\partial E}{\partial \phi} &= -\frac{\partial^2 C}{\partial E \partial S} < 0 \\
\frac{\partial E}{\partial Z} &= -\frac{\partial^2 C}{\partial E \partial Z} > 0 \\
\frac{\partial E}{\partial P} &= \frac{1}{\partial^2 E} > 0
\end{align*}
\]  
(3.16)

Upon substitution of 3.15 into 3.13 and 3.14, one obtains the following system:

\[
\begin{align*}
\dot{\phi} &= r\phi + \frac{\partial C}{\partial S} (S, E(S, \phi; P, Z); Z) \\
\dot{S} &= -E(S, \phi; P, Z)
\end{align*}
\]  
(3.17)

To complete the problem, boundary conditions are necessary. The terminal conditions are replaced with the steady state equations, \(\dot{S} = 0\) and \(\dot{\phi} = 0\). The optimal state and costate arcs thus obtained from 3.17, \((S(t; \beta), \phi(t; \beta))\), where \(\beta\) denotes the set of parameter values \((P, r, Z\) and \(S_0)\), can be plugged back into 3.17, producing:

\[
\begin{align*}
\dot{\phi}(t; \beta) &= r\phi(t; \beta) + \frac{\partial C}{\partial S} (S(t; \beta), E(S(t; \beta), \phi(t; \beta); P, Z); Z) \\
\dot{S}(t; \beta) &= -E(S(t; \beta), \phi(t; \beta); P, Z)
\end{align*}
\]  
(3.18)

The variational differential equations are derived from 3.18 and from the boundary conditions, by differentiating with respect to \(Z\). The result, using 3.16, is\(^{11}\):

\[
\begin{align*}
\frac{\partial \dot{\phi}}{\partial Z} &= r \frac{\partial \phi}{\partial Z} + \left(\frac{\partial^2 C}{\partial E^2} \frac{\partial S}{\partial Z} - \frac{\partial^2 C}{\partial E \partial Z} \left(\frac{\partial \phi}{\partial Z} + \frac{\partial^2 C}{\partial E \partial Z}\right) + \frac{\partial^2 C}{\partial E \partial S}\right) \\
\frac{\partial \dot{S}}{\partial Z} &= \frac{\partial^2 C}{\partial S^2} \frac{\partial S}{\partial Z} + \frac{1}{\partial^2 E} \left(\frac{\partial \phi}{\partial Z} + \frac{\partial^2 C}{\partial E \partial Z}\right) \\
\frac{\partial S_0}{\partial Z} &= 0
\end{align*}
\]  
(3.19)

as well as equations 3.6 and 3.7.

All derivatives are evaluated at \(\beta = \beta_0\), where \(\beta_0\) denotes the initial set of parameter values. The solution to 3.19, \(\left(\frac{\partial \phi}{\partial Z}, \frac{\partial S}{\partial Z}\right)\), provides the impact of changes

\(^{11}\)Notation has been simplified for clarity, but it should be remembered that from now on the values presented are optimal ones.
in $Z$ on the entire optimal paths of scarcity rent and stock. Instead of studying this general case, a distinction will once again be made between extraction and depletion-biased technological change.

- In the extraction-biased case, 3.19 becomes:

$$
\frac{\partial \phi}{\partial Z} = r \frac{\partial \phi}{\partial Z} + \left( \frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 C}{\partial E \partial Z} \right) \frac{\partial S}{\partial Z} - \frac{\partial^2 C}{\partial E^2} \left( \frac{\partial \phi}{\partial Z} + \frac{\partial^2 C}{\partial E \partial Z} \right) \frac{\partial \phi}{\partial Z} + \frac{\partial^2 C}{\partial E \partial Z} \frac{\partial \phi}{\partial Z} \frac{\partial S}{\partial Z} + \frac{1}{\partial E^2} \left( \frac{\partial \phi}{\partial Z} + \frac{\partial^2 C}{\partial E \partial Z} \right) \frac{\partial S}{\partial Z} = 0
$$

(3.20)

as well as equations 3.8 and 3.9

Using cost function properties, a study of variable movements through the construction of a diagram in $\left( \frac{\partial \phi}{\partial Z}, \frac{\partial S}{\partial Z} \right)$ can be undertaken. Figure 3.1 represents such a diagram, using simplified notation\footnote{For the diagram it is assumed that $\frac{\partial^2 C}{\partial E \partial Z}$ does not change with stock or extraction, allowing one to plot the line $\frac{\partial \phi}{\partial Z} = - \frac{\partial^2 C}{\partial E \partial Z}$.}. The arrows indicate motion and are drawn using the following procedure: a sign for $\left( \frac{\partial \phi}{\partial Z}, \frac{\partial S}{\partial Z} \right)$ is assumed, then used to calculate the corresponding sign for $\left( \frac{\partial \phi}{\partial Z}, \frac{\partial S}{\partial Z} \right)$, which is then used to draw the appropriate motion arrows. The results obtained are summarized in Proposition 1.
Proposition 1. In the case of an extraction-biased technological change, that is, \( \frac{\partial^2 C}{\partial E \partial Z} < 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} = 0 \), where \( dZ > 0 \),

1. \( \frac{\partial S}{\partial Z} < 0 \) and \( 0 < \frac{\partial \phi}{\partial Z} < -\frac{\partial^2 C}{\partial E \partial Z} \)
2. \( \frac{\partial S}{\partial Z} < 0 \) and \( \frac{\partial \phi}{\partial Z} > 0 \forall t \in ]0, +\infty[ \)
3. \( 0 < \frac{\partial \phi}{\partial Z} < -\frac{\partial^2 C}{\partial E \partial Z} \) at \( t = 0 \), hence \( \frac{\partial S}{\partial Z} < 0 \), which implies \( \frac{\partial E}{\partial Z} > 0 \), i.e. initial extraction increases.

Proof: Equations 3.8 and 3.9 are used directly to obtain the steady state results. This defines an area in Figure 3.1. Looking at the motion arrows and recalling that \( \frac{\partial S}{\partial Z} = 0 \) at the initial moment, if the steady state is to be reached then necessarily \( \frac{\partial \phi}{\partial Z} > 0 \forall t \in ]0, +\infty[ \) and \( \frac{\partial S}{\partial Z} < 0 \forall t \in ]0, +\infty[ \). For \( t = 0 \), plugging \( \frac{\partial S}{\partial Z} = 0 \) and \( \frac{\partial \phi}{\partial Z} > 0 \) into the \( \frac{\partial \dot{S}}{\partial Z} \) equation in 3.20 provides the result.

All subsequent Propositions derive from using the same approach.

- For the depletion-biased case, 3.19 turns into:

\[
\frac{\partial \phi}{\partial Z} = \left( r - \frac{\partial^2 C}{\partial E \partial S} \right) \frac{\partial \phi}{\partial Z} + \left( \frac{\partial^2 C}{\partial S \partial^2} - \frac{\partial^2 C}{\partial E \partial Z} \right) \frac{\partial S}{\partial Z} + \frac{\partial^2 C}{\partial S \partial Z} \]

\[
\frac{\partial S}{\partial Z} = \frac{\partial^2 C}{\partial E \partial S} \frac{\partial \phi}{\partial Z} + \frac{1}{\partial^2 C} \frac{\partial \phi}{\partial Z} \]

\[
\frac{\partial S}{\partial Z} = 0
\]

as well as equations 3.10 and 3.11

Using a procedure analogous to the one described above, this case is pictured in Figure 3.2 and the associated results are in Proposition 2.

Proposition 2. In the case of a depletion-biased technological change, that is, \( \frac{\partial^2 C}{\partial S \partial Z} > 0 \) and \( \frac{\partial^2 C}{\partial E \partial Z} = 0 \), where \( dZ > 0 \),

1. \( \frac{\partial S}{\partial Z} < 0 \) and \( \frac{\partial \phi}{\partial Z} < 0 \)
2. \( \frac{\partial S}{\partial Z} < 0 \forall t \in ]0, +\infty[ \)
3. \( \frac{\partial \phi}{\partial Z} < 0 \) at \( t = 0 \), hence \( \frac{\partial S}{\partial Z} < 0 \) and once more initial extraction increases.
3.2. Single dZ and dP

In this section, the combined effect of a single change in technology (dZ) and in the price of the good (dP) is examined in a comparative dynamics perspective. The case of a single change in the price of the good (dP) was already examined in Caputo [2]. Except for the transversality conditions, and, consequently, for what occurs at the boundary, the motion diagram is the same as in Caputo [2]. Using these results, and those obtained in the previous section, the two special cases for technological change are analyzed. The comparative statics results are as follows:

- Extraction-biased technological change: \( \frac{\partial^2 C}{\partial E \partial Z} < 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} = 0 \)

In this case equations 3.6 and 3.7 become:

\[
\begin{align*}
\frac{d\bar{S}}{dZ} &= -\frac{r}{r} \frac{\partial^2 C}{\partial E \partial Z} \frac{\partial^2 C}{\partial E \partial S} dZ + \frac{r}{r} \frac{\partial^2 C}{\partial E \partial S} dP \\
\frac{d\bar{\phi}}{dZ} &= \frac{\partial^2 C}{\partial E \partial S} \frac{\partial^2 C}{\partial S^2} dZ - \frac{\partial^2 C}{\partial E \partial S} \frac{\partial^2 C}{\partial S^2} dP
\end{align*}
\]

(3.22)

(3.23)

Note that when \( dZ > 0 \) and \( dP > 0 \), the impact both on steady state scarcity rent and stock is amplified, as it is obtained by adding up the separate effects of the same sign. Technological improvement and price increase affect the problem in a similar manner, since both contribute to make extraction more attractive and stock more valuable. Therefore, the steady state stock decreases by more, and the steady state scarcity rent also
increases by more. However, when \( dZ > 0 \) and \( dP < 0 \), the signs both on \( d\bar{S} \) and on \( d\bar{\phi} \) are indetermined.

- Depletion-biased technological change: \( \frac{\partial^2 C}{\partial E \partial Z} = 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} > 0 \)

Now the relevant equations are:

\[
\begin{align*}
    d\bar{S} &= \frac{\partial^2 C}{\partial S \partial Z} dZ + \frac{r}{\partial^2 C \partial E^2 - \partial^2 C \partial S^2} \partial \frac{\partial^2 C}{\partial S \partial Z} dP \\
    d\bar{\phi} &= -\frac{\partial^2 C}{\partial E \partial S} dZ - \frac{\partial^2 C}{\partial E \partial S} \partial \frac{\partial^2 C}{\partial E^2} \partial \frac{\partial^2 C}{\partial S^2} dP
\end{align*}
\]

(3.24) (3.25)

This type of technological change lowers costs but also lowers scarcity rent, whereas a price increase lowers costs but raises stock value. Thus, if \( dZ > 0 \) and \( dP > 0 \) there is an additional decrease of steady state stock, while steady state scarcity rent may increase or decrease. On the contrary, when \( dZ > 0 \) and \( dP < 0 \), the impact on the steady state stock is not clear, while the steady state scarcity rent decreases by more.

Proceeding similarly as in section 3.1, comparative dynamics results are the following:

- In the extraction-biased case, the variational differential equations become:

\[
\begin{align*}
    d\dot{\phi} &= r d\phi - \frac{\partial^2 C}{\partial E^2} (d\phi + \frac{\partial^2 C}{\partial E \partial Z} dZ - dP) + \left( \frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 C}{\partial E \partial S} \right) dS \\
    d\dot{S} &= \frac{\partial^2 C}{\partial S^2} dS + \frac{1}{\partial^2 C \partial E^2} (d\phi - dP + \frac{\partial^2 C}{\partial E \partial Z} dZ) \\
    dS_0 &= 0
\end{align*}
\]

(3.26)

as well as equations 3.22 and 3.23

Using cost function properties, a study of variable movements through the construction of a diagram in \( d\phi \) and \( dS \) can now be undertaken. Again, the arrows indicate motion and are drawn using the following procedure: a sign for \( d\phi \) and \( dS \) is assumed, then used to calculate the corresponding sign for \( d\dot{\phi} \) and \( d\dot{S} \), which is then used to draw the appropriate motion arrows. Figure 3.3 depicts the case of a rise in price and Figure 3.4 represents the first possibility discussed in Proposition 4 for a fall in price. As usual, all results are summarized in the following Propositions.

---

13 From now on, \( d\phi = \frac{\partial \phi}{\partial Z} dZ + \frac{\partial \phi}{\partial P} dP \), and \( dS = \frac{\partial S}{\partial Z} dZ + \frac{\partial S}{\partial P} dP \). Likewise for \( d\dot{\phi} \) and \( d\dot{S} \).
Figure 3.3: Extraction-biased change with a price increase

Proposition 3. In the case of an extraction-biased technological change, that is, \( \frac{\partial^2 C}{\partial E \partial Z} < 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} = 0 \), where \( dZ > 0 \) and \( dP > 0 \), then,

1. \( dS < 0 \) and \( 0 < \phi < dP - \frac{\partial^2 C}{\partial E \partial Z} dZ \)
2. \( dS < 0 \) and \( \phi > 0 \ \forall t \in [0, +\infty] \)
3. \( 0 < d\phi < dP - \frac{\partial^2 C}{\partial E \partial Z} dZ \) at \( t = 0 \), hence \( d\dot{S} < 0 \) and \( dE > 0 \), ie, initial extraction increases.

Note that Figures 3.1 and 3.3 look the same except for the ridge line. This is to be expected given the similar eœects of extraction-biased technological improvement and price increase.

Proposition 4. In the case of an extraction-biased technological change, that is, \( \frac{\partial^2 C}{\partial E \partial Z} < 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} = 0 \), where \( dZ > 0 \) and \( dP < 0 \), several possibilities can occur:

- (i) if \( \frac{\partial^2 C}{\partial E \partial Z} dZ - dP > 0 \), then:
  1. \( dS > 0 \) and \( -(\frac{\partial^2 C}{\partial E \partial Z} dZ - dP) < \phi < 0 \)
  2. \( dS > 0 \), \( d\phi < 0 \ \forall t \in [0, +\infty] \)
  3. \( -(\frac{\partial^2 C}{\partial E \partial Z} dZ - dP) < d\phi < 0 \) at \( t = 0 \), thus \( d\dot{S} > 0 \) and initial extraction decreases.
- (ii) if \( \frac{\partial^2 C}{\partial E \partial Z} dZ - dP < 0 \), then:
Figur e 3.4: Extraction-biase d change with a pric e decreas e (cas e (i))

1. \( dS < 0 \) and \( 0 < d\phi < -(\frac{\partial^2 C}{\partial E \partial Z}dZ - dP) \)
2. \( dS < 0, \ d\phi > 0, \ \forall t \in [0, +\infty] \)
3. \( 0 < d\phi < -(\frac{\partial^2 C}{\partial E \partial Z}dZ - dP) \) at \( t = 0 \), thus \( d\dot{S} < 0 \) and initial extraction increases.

The diagram for this case also looks exactly the same as that in Figure 3.1, except for the ridge line, hence it is not presented here. Intuitively, this means a small enough price decrease only dampens the effects of technological change. The qualitative results are exactly the same as in Proposition 1.

(iii) if \( \frac{\partial^2 C}{\partial E \partial Z}dZ - dP = 0 \), then the two effects cancel out, and in diagrammatic terms we would never leave the origin.

- For the depletion-biased case, the variational differencial equations are:

\[
\begin{align*}
\dot{d}\phi &= r\phi + \left(\frac{\partial^2 C}{\partial S^2} - \frac{\left(\frac{\partial^2 C}{\partial E \partial S}\right)^2}{\partial^2 C/\partial E^2}\right) dS - \frac{\partial^2 C}{\partial E \partial S}(d\phi - dP) + \frac{\partial^2 C}{\partial S \partial Z}dZ \\
\dot{d}\hat{S} &= \frac{1}{\partial^2 C/\partial E^2} (d\phi - dP) + \frac{\partial^2 C}{\partial E \partial S}dS \\
dS_0 &= 0
\end{align*}
\]

(3.27)
as well as equations 3.24 and 3.25

Using Figures 3.5 and 3.6 respectively, Propositions 5 and 6 are obtained.
Proposition 5. In the case of a depletion-biased technological change, that is, \( \frac{\partial^2 C}{\partial S \partial Z} > 0 \) and \( \frac{\partial^2 C}{\partial E \partial Z} = 0 \), where \( dZ > 0 \) and \( dP > 0 \), then:

1. (i) \( dS < 0 \) and \( d\phi < 0 \), or (ii) \( dS < 0 \) and \( 0 < d\phi < dP \)
2. \( dS < 0 \ \forall \ t \in ]0, +\infty[ \)
3. \( d\phi < dP \) at \( t = 0 \), hence \( d\dot{S} < 0 \), and once more initial extraction increases.

Proposition 6. In the case of a depletion-biased technological change, that is, \( \frac{\partial^2 C}{\partial S \partial Z} > 0 \) and \( \frac{\partial^2 C}{\partial E \partial Z} = 0 \), where \( dZ > 0 \) and \( dP < 0 \), then:
1. (i) $dS < 0$ and $d\phi < dP < 0$, or (ii) $dS > 0$ and $dP < d\phi < 0$
2. $dS$ and $d\phi$ anywhere except in the first quadrant, $\forall t \in ]0, +\infty[$
3. the sign for $d\dot{S}$ at $t = 0$ is uncertain, so initial extraction may increase or decrease.

3.3. The Impact on Scarcity Measures

3.3.1. Single dZ

The purpose of this section is to study which combinations of scarcity measures’ movements are compatible with optimal behaviour, when there is a single technological improvement, and assuming that price is exogenous and invariant. This assumption will be relaxed below.

If there is a single technological improvement, the comparative dynamics results obtained in the previous section can be combined with equations 2.10 and 2.11 to describe possible optimal paths for scarcity measures, namely scarcity rent and marginal extraction cost (price is still assumed constant at this point). To begin with, note that the conclusions of section 2.1 can be used in the analysis of the two separate optimal solutions (one for $Z$ and another for $Z'$); the aforementioned equations hold for both of them. Hence, in either case scarcity rent is generally growing, and marginal extraction cost is decreasing (even though the exact amplitude of these movements cannot be compared). The next step is to consider what happens to the optimal path if technology is initially at $Z$ and then increases to $Z'$.

In the extraction-biased case, it was shown in Proposition 1 that scarcity rent increases with technology. Considering that the price is constant, this implies that marginal extraction cost becomes lower, $\frac{d(\frac{\partial C}{\partial E})}{dz} < 0 \ \forall t$. These effects have the same sign as when technology is invariant, therefore it can be concluded that the optimal path when there is a technological change is also characterized by $\dot{\phi} > 0$ and $\frac{\partial C}{\partial E} < 0$. Extraction-biased technological change does not alter the direction of scarcity measure behavior.

For depletion-biased technological change, on the contrary, the nature of the optimal path is changed. Proposition 2 shows that both initial$^{14}$ and steady state scarcity rent decrease with technological change. Hence, scarcity rent cannot be growing monotonically, and likewise marginal extraction cost cannot always be decreasing.

$^{14}$Here, initial means at the moment when technology changes.
It must be emphasized that, unlike what is done in Farzin [6], for a competitive producer the price path should be the starting point and not the result of other scarcity measure movements. That is the perspective of this paper; exogenous changes in price are analyzed in the following sections.

3.3.2. Single dZ and dP

In the presence of a single technological improvement and a single change in price, the consequences for scarcity measures’ movements are now examined. Emphasis is placed on ruling out certain behaviours, namely monotonical ones, by proving them to be incompatible with optimal decisions.

In the extraction-biased case, for \( dP > 0 \) it was shown in Proposition 3 that both initial and steady state scarcity rent increase. Therefore, scarcity rent cannot decrease monotonically with marginal extraction cost always increasing. That is, \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} > 0 \) cannot characterize the optimal path\(^{15} \). When \( dP < 0 \), by Proposition 4, in case (i), both initial and steady state scarcity rent decrease. Therefore, scarcity rent cannot increase monotonically with marginal extraction cost always decreasing. In case (ii), initial and steady state scarcity rent increase. Therefore, scarcity rent cannot decrease monotonically, whether marginal extraction cost is always increasing, decreasing or constant. For case (iii), scarcity rent can either increase or decrease.

In the case of a depletion-biased technological change, for \( dP > 0 \), by Proposition 5, all paths are potential optimal paths. For \( dP < 0 \), Proposition 6 shows that initial and steady state scarcity rents decrease. Thus, scarcity rent cannot increase monotonically with marginal extraction cost also decreasing monotonically. These results can be summarized as follows:

- In the extraction-biased case, the following paths cannot be optimal:
  
  \[ dP > 0: \]
  
  * \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} > 0 \);  
  
  \[ dP < 0: \]
  
  * (i) \( \dot{\phi} > 0 \) and \( \frac{\partial \dot{C}}{\partial E} < 0 \);  
  
  * (ii) \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} < 0 \); \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} > 0 \); \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} = 0 \);  

\(^{15}\)The cases where \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} < 0 \) or \( \dot{\phi} < 0 \) and \( \frac{\partial \dot{C}}{\partial E} = 0 \) are immediately ruled out by equation 2.6.
In the depletion-biased case, the following paths cannot be optimal:

\[ dP < 0: \]

* (i) \( \dot{\phi} > 0 \), and \( \frac{\partial C}{\partial E} < 0 \);

* (ii) \( \dot{\phi} < 0 \), and \( \frac{\partial C}{\partial E} > 0 \); \( \dot{\phi} < 0 \), and \( \frac{\partial C}{\partial E} < 0 \); \( \dot{\phi} < 0 \), and \( \frac{\partial C}{\partial E} = 0 \).

Thus, for \( dZ > 0 \) and \( dP > 0 \), or \( dP < 0 \), monotonic optimal paths for the scarcity rent, and, hence, to the marginal extraction cost are only excluded in the case of a depletion-biased technological change, for \( dP < 0 \). In the case of an extraction-biased technological change, it can be optimal to have the scarcity rent increasing monotonically.

4. Endogenous price

Allowing price to be determined by market equilibrium requires the introduction of a demand function. Here it will be assumed that producers are competitive, so that the model described in the previous sections is an accurate description of resource supply behavior. However, although price is taken as given by each producer, price behavior will be determined by demand conditions.

For each period, the total quantity demanded of the resource will be equal to the sum of the quantities offered by each individual producer, \( Q_t = \int_1^N E_i(t)di \) (where \( i \) is the producer). Demand for the resource is assumed to be a continuous function of price at \( t \), \( Q^D(P_t, t) \), which can inverted to obtain \( P(Q_t, t) \). There is a finite choke price in every period, ie. there is a price for which \( Q^D = 0 \). Market equilibrium is given by:

\[
\begin{align*}
Q_t &= \int_1^N E_i(t)di = Q^D(P_t, t) \quad \forall t \\
\int_0^T Q^D(P_t, t) &= \int_0^T \int_1^N E_i(t)di \leq \int_1^N S_i(t)di = S
\end{align*}
\] (4.1)

The first equation is a static equilibrium condition, whereas the second one is a dynamic equilibrium condition. The existence of an equilibrium can be demonstrated using cost function convexity (see Sweeney [11, pp.819-23]).
4.1. Equilibrium without technological improvements

Assuming an infinite horizon, each producer $i$ will behave according to the first order conditions of problem 2.1, namely:

\[
\begin{align*}
P_t &= \frac{\partial C_i}{\partial E_t} + \phi^i \quad i f \quad E_t^i > 0 \\
\dot{P}_t &= \frac{\partial C_i}{\partial E_t} + \phi^i \quad i f \quad E_t^i = 0 \\
\dot{\phi}^i &= \phi^i r + \frac{\partial C_i}{\partial S_t} \\
\dot{S}_t^i &= -E_t^i \\
\lim_{t \to \infty} \phi_t^i e^{-rt} S_t^i &= 0 \quad \lim_{t \to \infty} \phi_t^i \geq 0 \quad \lim_{t \to \infty} S_t^i \geq 0
\end{align*}
\]  

(4.2)

If demand is constant or growing, prices will necessarily grow as long as the resource is being extracted. This result can be obtained recalling equations 2.7, 2.9 and 2.11. Since there is no technological change:

\[
\dot{P} = \frac{\partial^2 C_i}{\partial E^2} \dot{E} + \frac{\partial^2 C_i}{\partial E \partial S} \dot{S} + \dot{\phi}^i
\]  

(4.3)

If $\dot{P} < 0$ and demand is not decreasing, $\dot{E} > 0$ (for the resource market as a whole, and, since all producers are alike, for each individual deposit as well). Equation 4.3 can only be verified if $\dot{\phi}^i < 0$, which is incompatible with $\frac{\partial C_i}{\partial E} > 0$ (see section 2.1). If demand is actually constant, growing prices imply decreasing extraction. (more to come...)

A. Appendix

The representation of $\dot{S} = 0$ and $\dot{\phi} = 0$ in a phase diagram can be useful for comparative dynamics. The slope of $\dot{S} = 0$ is positive; it can be calculated implicitly using conditions 3.12 and 3.14:

\[
\frac{\partial \phi}{\partial S} = -\frac{\partial^2 C}{\partial E \partial S} > 0
\]  

(A.1)

Using equation 3.13, the slope of $\dot{\phi} = 0$ is:

\[
\frac{\partial \phi}{\partial S} = -\frac{\left(\frac{\partial^2 C}{\partial S^2} + \frac{\partial^2 C}{\partial E \partial S} \frac{\partial E}{\partial S}\right)}{r}
\]  

(A.2)
Substituting for $\frac{\partial E}{\partial S}$:

$$\frac{\partial \phi}{\partial S} = -\frac{\left(\frac{\partial^2 C}{\partial S^2} \frac{\partial C}{\partial E} - \left(\frac{\partial C}{\partial E}\right)^2\right)}{\frac{\partial^2 C}{\partial E^2}}$$

(A.3)

With cost function convexity, this expression is negative.

The results attained so far are represented in Figure A.1. The optimal path has been drawn assuming monotonicity of $\frac{\partial C}{\partial S}$, hence the negative slope (it was shown in section 2.1 that $\phi$ should be growing over time).

If $dZ > 0$, then the $S$-constant locus will be shifted upward, by equation 3.12:

$$d\phi = -\frac{\partial^2 C}{\partial E \partial Z} dZ > 0$$

(A.4)

As for the $\phi$-constant locus, it may be shifted upward or downward. Differentiation of equations 3.12 and 3.13 yields:

$$dE = -\frac{\partial^2 C}{\partial E \partial S} dZ + d\phi$$

(A.5)

$$rd\phi = -\frac{\partial^2 C}{\partial E \partial S} dE - \frac{\partial^2 C}{\partial S \partial Z} dZ$$

(A.6)

Combining these two equations and simplifying:

$$d\phi = \frac{\partial^2 C}{\partial E \partial S} \frac{\partial^2 C}{\partial E \partial Z} - \frac{\partial^2 C}{\partial S \partial Z} \frac{\partial^2 C}{\partial E^2} dZ$$

(A.7)
The sign of this expression is indeterminate. Following what was done in the paper, two cases can be distinguished: extraction-biased and depletion-biased technological change. Clearly, if \( \frac{\partial^2 C}{\partial E \partial Z} < 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} = 0 \), the \( \dot{\phi} = 0 \) locus will be shifted upward, and if \( \frac{\partial^2 C}{\partial E \partial Z} = 0 \) and \( \frac{\partial^2 C}{\partial S \partial Z} > 0 \) it will be shifted downward, providing the steady state results that were previously obtained. The phase diagrams are presented in Figures A.2 and A.3 respectively (note that the \( \dot{S} = 0 \) locus does not shift for the depletion-biased case, since the marginal cost of extraction is not directly affected; as for the extraction-biased case, it can be shown that the upward shift of the \( \dot{S} = 0 \) locus is larger than that of the \( \dot{\phi} = 0 \) locus).
References


$d\phi = dP$
\[ d\phi = dP - C_{EZ} dZ \]

fig 3

\[ d\phi = dP \]

fig 4
\[ d\Phi = dP - C_{EZ} dZ \]

fig. 5

\[ \Phi_z = -C_{EZ} \]

fig. 6
\[ \phi = 0 \]

\[ S = 0 \]

fig. 9