IRREVERSIBILITY, POLLUTANT STOCK UNCERTAINTY, 
AND THE TIMING OF IMPLEMENTATION OF EMISSION LIMITS

by

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ABSTRACT

Using real options, we analyze the timing of implementing emission limits for a decaying pollutant when random environmental effects affect its stock. Two types of irreversibility are present: the first type results from long-term environmental damage, and the second from sunk costs in pollution abatement. We assume that social damage from pollution is a power function of the stock of pollutant. With reference to the deterministic case, we find that a small level of uncertainty may either delay or advance an investment designed to cut pollutant emissions, depending on the cost of reducing emissions relative to expected social costs. When environmental uncertainty increases, environmental damages end up prevailing and pollutant emissions should be curbed immediately. More generally, this paper shows that a deterministic model is not always appropriate for dealing with stochastic environmental problems, even when uncertainty is small.

I. INTRODUCTION

The impact of uncertainty and irreversibility on environmental policy has long been a subject of debate among economists and policy makers, and it has not yet been resolved. In the context of a stock pollutant, irreversibility has two basic sources. First, damage to the environment may be so high that it becomes permanent or the amount of pollution may be so large that it leads to long-term damages. Second, investments to control or reduce a stock externality are often at least partially sunk. An example is the installation of scrubbers by a utility. In general, there are two aspects to uncertainty, which are not always distinguished: one aspect is risk aversion, which varies with each economic agent, and the other is the arrival of information over time. The resources devoted to better understand environmental issues (e.g., global climate change, the
preservation of biodiversity, or acid rain, to name a few) highlight the pervasive nature and importance of uncertainty in that environmental problems.

The combined effect of irreversibility and uncertainty led Weisbrod (1964) to introduce the concept of option value. Using the case of a national park where revenues do not cover total costs, he argues that the value of the possibility (“the option”) to choose the timing of a decision that has irreversible consequences should be included in a cost-benefit analysis. Weisbrod’s pioneering work, has given rise, however, to differing interpretations of the nature of option value, its sign, and its magnitude. Some authors, following Cicchetti and Freeman (1971) or Schmalensee (1972), view option value as a risk premium paid by risk averse consumers to reduce the impact of uncertainty in the supply of an environmental good. Others, following the interpretation of Arrow and Fisher (1974) and Henry (1974), prefer to talk of quasi-option value to describe the welfare gains associated with delaying a decision which is at least partly irreversible and involves some uncertainty about the payoffs of alternative choices. For them, quasi-option value is akin to a tax on development, which corrects benefits from development in a traditional cost-benefit analysis. This tax can also be a subsidy, depending on the source of uncertainty and the nature of irreversibilities. With this interpretation, quasi-option value can be linked to the arrival of information over time, but it is independent from risk aversion (see Fisher and Hanemann (1987) for a summary of the interpretations of option and quasi-option value).

Many papers in the quasi option literature are based on two-period discrete time models. One standard result (e.g., see Arrow and Fisher, 1974) is that, in the presence of environmental irreversibility, a standard cost-benefit analysis is biased against
Freixas and Laffont (1980) generalize this result, but Epstein (1980) show that it is not always valid. More recently, Kolstad (1996), in a study of pollution stock effects and sunk emission control capital shows that there is an irreversibility effect only when there is excess stock pollutant. The generality of results based on two-period discrete time models has been questioned by Hanemann (1989).

The lack of consensus on how to define and measure option value has hampered its inclusion in actual cost-benefit analyses, and it has led some authors to question its usefulness (Freeman, 1993). To avoid this controversy while keeping Weisbrod’s original intuition, we rely instead on the theory of real option, which has its roots in finance. In our context, we can define a real option as the value of flexibility with regard to a decision with irreversible environmental and economic consequences. For a good introduction to the real options approach in problems of investments under uncertainty, see Dixit and Pindyck (1994).

In this paper, we revisit the problem considered by Kolstad using an approach similar to Pindyck’s (in Dixit and Pindyck, 1994). We are concerned with the tension between irreversible environmental effects and sunk investments to control pollutant emissions when the stock of pollutant varies randomly. Instead of a two-period discrete time model, we use a more general continuous time model. To obtain manageable close-form results, however, we adopt specific functional forms for our social damage function (a power function of the stock of pollutant) and for the diffusion process followed by the stock of pollutant.

This paper is organized as follows. In Section II, we introduce a simple continuous-time model, which features a single stock externality. We formulate our
pollution control problem as a stopping problem in which a risk neutral social planner has to choose when and how much to invest in a one-time reduction in the rate of pollutant emission. In Section III, we solve the corresponding deterministic problem to get a benchmark for the impact of randomness in the stock of pollutant on the timing of reducing pollutant emissions. This permits us to illustrate the concept of option value from the point of view of real options. In Section IV, we analyze the stochastic case. To explore a range of stochastic behaviors, we consider two stochastic models. A numerical application is provided in Section V to illustrate our results. The last section summarizes our conclusions.

II. A MODEL OF POLLUTANT STOCK UNCERTAINTY

We consider a stylized model with one environmental pollutant, which decays at rate $\alpha > 0$. We note $X$ the stock of this pollutant and $E_1$ its rate of emission. To focus solely on the variability of $X$, we assume that $E_1$ is constant. Problems with more than one stochastic variable are notoriously difficult to solve analytically. Because of the randomness of physical and chemical processes that contribute to the decay of the pollutant, we suppose that $X$ follows a diffusion process, which can be written:

$$dX = (E_1 - \alpha X)dt + \sigma(X)dz$$

Let $x_{ic} = \frac{E_1}{\alpha}$. To obtain explicit solutions and examine their variations, we consider two particular forms of Equation (1):

$$dX = \alpha(x_{ic} - X)dt + \sqrt{X^\rho}dz, \rho = 1,2$$
The quantity \( vX^p \geq 0 \) is the infinitesimal variance of the stock pollutant process and \( dz \) is an increment of a standard Wiener process (for an introduction to stochastic calculus for economists, see Dixit, 1993). When \( \rho = 1 \) (Model 1), the infinitesimal variance of the process followed by \( X \) increases linearly with \( X \), whereas when \( \rho = 2 \) (Model 2), it increases with the square of \( X \), just like for the geometric Brownian motion. A model with a stochastic stock of pollutant with a fixed infinitesimal variance does not seem reasonable in this context. From Equation (2), we see that \( X \) remains non-negative and tends to revert to \( x_{1c} \); thus, the decay rate, \( \alpha \), also characterizes the speed of reversion.

The initial stock of pollutant, which is known, is noted \( X(0) \).

We further assume that the flow of social costs resulting from pollution damage, noted \( C_m(X) \), is given by:

\[
C_m(X) = -cX^m
\]

(3)

The constant \( c > 0 \) is a valuation parameter and \( m \geq 2 \) is an integer, so \( C_m \) is strictly convex.

Pollutant emissions can be decreased from \( E_1 \) to a constant \( E_2 < E_1 \), at a cost \( K \), which may depend on \( E_1 \) and on \( E_1 - E_2 \). We suppose that \( K \) is completely sunk, which is often reasonable for pollution control measures (e.g., the installation of scrubbers by utilities). After the emission reduction investment is made, \( X \) follows the new process:

\[
dX = \alpha(x_{2c} - X)dt + \sqrt{vX^p} \, dz, \, \rho = 1, 2
\]

(4)

where \( x_{2c} = \frac{E_2}{\alpha} \).

To eliminate the impact of risk aversion, we consider a risk-neutral social planner. Her goal is to find \( E_2 \) (\( 0 \leq E_2 \leq E_1 \)), the rate to which pollutant emission should be reduced,
and $T$, the socially optimal time of cutting pollutant emissions to $E_2$, in order to minimize the present value function:

$$J(T,E_2) = \mathbb{E}_0 \int_0^\infty cX^m e^{-rt} dt + e^{-rT}K(E_2,E_1 - E_2)$$

subject to Equation (2) for $0 \leq t \leq T$ and to Equation (4) for $t > T$, given $X(0)$. Here, $\mathbb{E}_0$ is the expectation operator for information available at time $t=0$, and $r$ is the social discount rate.

This optimization problem can be solved in two steps. First, for an arbitrary value of $E_2$, such that $0 \leq E_2 < E_1$, we calculate the critical stock of pollutant, noted $x^*$, at which the rate of pollutant emission should be reduced from $E_1$ to $E_2$. Once $x^*$ is known, we can calculate $\mathbb{E}_0 T(x^*;E_2)$, the expected time at which the stock of pollutant reaches $x^*$ for the first time given an initial stock of $X(0)$. When $v > 0$, $X$ is a random variable and $T(x^*;E_2)$ is a stopping time. For $E_2$ fixed, $X < x^*$ defines the so-called “continuation region,” or region 1, where the optimal decision is to wait. As soon as $X \geq x^*$, which defines the so-called “stopping region,” or region 2, the rate of pollutant emissions should be reduced to $E_2$. The second step consists of finding the value of $E_2$, noted $E_2^*$, that will maximize the objective function.

In this paper, we want to analyze the impact of $v$ on the timing of the decision to make a sunk investment $K$ to reduce pollution emissions, for arbitrary functional forms of $K$. Hence, we focus on the determination of $x^*$ as a function of $v$ for a fixed $E_2$, and we ignore the determination of the optimal value of $T$ which comes into play in the determination of $E_2$. 
This is a standard stopping problem that bears similarities with an optimal investment problem. To solve, we use concepts from the theory of real options and stochastic dynamic programming. We note $V_{i,m}(x;v)$ the value function in region “i”. The corresponding Hamilton-Jacobi-Bellman equation is:

$$
(6) \quad rV_{i,m}(x;v) = -cx^m + \alpha(x - E_i) \frac{dV_{i,m}(x;v)}{dx} + \frac{\sigma^2}{2} \frac{d^2V_{i,m}(x;v)}{dx^2}, \quad i = 1, 2
$$

The left side of Equation (6) can be interpreted as a return; the first term on the right side is the flow of social pollution costs; and the last term is the capital gains term which results from Ito’s lemma.

Equation (6) is a second-order linear equation. Its solution is the sum of a particular solution, noted $P_{i,m}(x;v)$, plus the general solution of the associated homogeneous equation, noted $\phi(x;v)$. We select $P_{i,m}(x;v)$ so that it represents the expected social costs from emitting pollution at rate $E_i$ forever, given $x$, the current stock of pollutant. $\phi(x;v)$ is the value of the option to choose the timing for reducing emissions. Since it represents the value of the possibility to do something, it is non-negative. In this context, waiting reduces the present value of the cost of implementing the proposed policy while reducing emissions earlier reduces the present value of pollution damages. In financial terms, $\phi(x;v)$ is a perpetual American option.

When we consider a one-time reduction in pollutant emissions, there is no option term after pollutant emissions have been reduced to $E_2$. Thus, the solutions of Equation (6) in regions 1 and 2 are respectively:

$$
(7) \quad V_{1,m}(x;v) = \phi(x;v) + P_{1,m}(x;v) \quad V_{2,m}(x;v) = P_{2,m}(x;v)
$$
To find \( x^* \), we need two additional conditions (for a heuristic proof of the necessity and sufficiency of these conditions, see Dixit and Pindyck, 1994.) First, at \( x^* \), the value of the option to reduce the rate of pollutant emissions plus the social cost of polluting forever at rate \( E_1 \) should equal the social cost of polluting forever at rate \( E_2 \) plus the cost of reducing emissions from \( E_1 \) to \( E_2 \). This is the continuity condition:

\[
\phi_1(x^*;v) + P_{1,m}(x^*;v) = P_{2,m}(x^*;v) - K
\]

The second condition, called "smooth-pasting," says that, when it is optimal to exercise the option to reduce emissions, the marginal change in the value of the option equals the marginal change in the difference of social pollution costs:

\[
\frac{d\phi(x^*;v)}{dx} = \frac{dP_{2,m}(x^*;v)}{dx} - \frac{dP_{1,m}(x^*;v)}{dx}
\]

By combining these two conditions, we obtain a "stopping rule": for this problem, it is an equation whose smallest non-negative root defines the critical stock of pollutant at which pollutant emissions should be reduced from \( E_1 \) to \( E_2 \).

III. SOLUTION OF THE DETERMINISTIC MODEL

In this section, \( v=0 \) so both stochastic models reduce to the same deterministic form. In that case, Equations (1), (4), and (6) simplify to first order linear differential equations. Integrating Equations (2) and (4), we obtain:

\[
X(t) = \begin{cases} 
    x_{1c} + (X(0) - x_{1c})e^{-\alpha t}, & 0 \leq t \leq T \\
    x_{2c} + (X(T) - x_{2c})e^{-\alpha(T-t)}, & t > T 
\end{cases}
\]

Thus, when \( E_i \) is held constant, \( X \) converges monotonically towards \( x_{ic} \). With this result, we can calculate the present value of social pollution costs.
LEMMA 1: If the rate of pollutant emission is fixed at $E_i$ and the initial stock of pollutant is $x$, the present value of social pollution costs is:

$$P_{i,m}(x;0) = -c \sum_{k=0}^{m} x^k \frac{m!}{k!} \prod_{i=k}^{m} \left( r + l \alpha \right), \quad i = 1, 2$$

Moreover, the option term is:

$$\varphi(X;0) = \text{Max} (\varphi(X), 0)$$

with

$$\tilde{\varphi}(X) = A_0 \left[ \alpha X - E_i \right]^{\frac{-1}{\alpha}}$$

$A_0$ is a non-negative constant to be determined jointly with the critical stock of pollutant $x_0^*$, at which it is optimal to reduce pollutant emission from $E_1$ to $E_2$.

Proof of LEMMA 1. Since we know how the stock of pollutant changes with time, we find the present value of social pollution costs by a direct calculation of

$$- \int_0^{+\infty} cX^2 e^{-rt} dt$$

subject to $X(t) = \left( x - \frac{E_i}{\alpha} \right) e^{-\alpha t} + \frac{E_i}{\alpha}$. The option term is the solution of the homogeneous equation associated to Equation (6) with $\nu=0$. To calculate $x_0^*$, we introduce Equations (11), (12) and (13) into the continuity and smooth-pasting conditions and take their ratio to get rid of the unknown parameter $A_0$. We find that $x_0^*$ is the smallest non-negative real that verifies:

$$\frac{E_1 - \alpha x}{r} = \frac{P_{2,m}(x;0) - P_{1,m}(x;0) - K}{P_{2,m}(x;0) - P_{1,m}(x;0)}$$
This is our deterministic stopping rule. At $x_0^*$, the net savings from reducing emissions from $E_1$ to $E_2$ divided by the marginal savings of reducing emissions, equals the ratio of the option term by its marginal value.

However, when $x_0^*$ is greater than $x_{1c}$, the smooth-pasting condition cannot be verified. Indeed, $\frac{d\bar{\varphi}(x)}{dx} = \frac{r}{E_1 - \alpha x} \bar{\varphi}(x)$, so the left-hand side of the smooth pasting condition, given by $\frac{d\bar{\varphi}(x)}{dx}$, is positive if $x<x_{1c}$ and negative if $x>x_{1c}$. From Equation (11), we see that $\forall x?0, \frac{dP_{2,m}(x;0)}{dx} - \frac{dP_{1,m}(x;0)}{dx} > 0$. This is the right-hand side of the smooth pasting condition. Hence, when $x_0^* > x_{1c}$, the option value is zero, so making a sunk investment to reduce pollutant emissions is a “now or never” proposition: we should invest now in pollution reduction if $J(0,E_2) > J(+\infty,E_2)$, and never otherwise.

When $m=2$, Equation (14) simplifies and we can derive an explicit expression for $x_0^*$:

$$x_0^* = \frac{r(2\alpha + r)K}{2c(E_1 - E_2)} - \frac{E_2}{\alpha + r}$$  

As expected, $x_0^*$ increases with $K$, $\alpha$, and $r$, and decreases when $c$ or $E_1$ increase. Also note that $x_0^*$ can be negative if the cost of switching from $E_1$ to $E_2$ is “low”; in that case, pollutant emissions should be reduced immediately. Once $x_0^*$ is known for all values of $E_2$ of interest, we can calculate the optimal level of pollution reduction $E_2^*$ by a simple optimization.
IV. ANALYSIS OF THE STOCHASTICAL MODELS

We now assume that \( v > 0 \). We start by calculating the expressions of expected social costs and the option terms.

**Lemma 2:** For Model 1 (\( \varphi = 1 \)), the expected social costs from continuing to pollute at rate \( E_i \) forever, given an initial stock \( x \), is:

\[
P_{i,m}(x;v) = -c \sum_{k=0}^{m} x^k \frac{m!}{k!} \prod_{l=0}^{m-1} \frac{(E_i + lv/2)}{(r + l\alpha)}, \quad i = 1, 2
\]

The corresponding option term is given by Equation (12) with:

\[
\tilde{\phi}(x;v) = A_0 \Phi \left( r, \frac{2E_i v \alpha}{\alpha}; 2 \frac{\alpha v^2}{\alpha} x \right)
\]

\( A_0 \) is a non-negative constant and \( \Phi(a,b;y) \) is the confluent hypergeometric function of the first kind with argument \( y \) and parameters \( a \) and \( b \).

For Model 2 (\( \varphi = 2 \)), the expected social costs from continuing to pollute at rate \( E_i \) forever, given an initial stock \( x \), is:

\[
P_{i,m}(x;v) = -c \sum_{k=0}^{m} x^k \frac{m!}{k!} \prod_{l=0}^{m-k} \frac{E_i^{m-k}}{(r + l\alpha - l(l-1)v/2)}, \quad i = 1, 2
\]

For \( P_{i,m}(x;v) \) to be finite, we must have:

\[
0 \leq v < 2 \frac{r + m\alpha}{m(m-1)}
\]

The corresponding option term is given by Equation (12) with:
\( \tilde{\phi}(x) = B_0 \left( \frac{E_1}{x} \right)^{\beta} \Psi \left( \beta, \zeta, \frac{2E_1}{\nu x} \right) \)

\( B_0 \) is a non-negative constant and \( \Psi(a,b;z) \) is the confluent hypergeometric function of the second kind with argument \( y \) and parameters \( a \) and \( b \). \( \beta \) and \( \zeta \) are functions of \( r, \nu, \) and \( \alpha \), defined by:

\[
\beta = \left( \frac{\nu}{2} + \alpha \right) + \sqrt{\left( \frac{\nu}{2} + \alpha \right)^2 + 2rv}, \quad \zeta = 2(\beta + 1) + \frac{2\alpha}{\nu}
\]

Proof of LEMMA 2. For both models, we derive the expected social costs from the moment generating function of \( X \), noted \( M(\theta,t) \), and from the relationship:

\[
\frac{\partial^n M(0,t)}{\partial \theta^n} = (-1)^n \mathcal{E}(x^n)
\]

Details of the calculations of \( M(\theta,t) \) are presented in Appendix A. Note that the absolute value of \( P_{i,m}(x;\nu) \) is increasing in \( \nu \). As in the deterministic case, \( P_{i,m} \) increases when \( r \) or \( \alpha \)

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1 It has the series representation: \( \Phi(a,b;z) = \sum_{k=0}^{+\infty} \frac{(a)_k z^k}{(b)_k k!} \), where \( (a)_k = a \cdot (a+1) \cdot \cdots \cdot (a+k-1) \).

2 It is defined by: \( \Psi(a,b;z) = \frac{\Gamma(1+b)}{\Gamma(1+a-b)} \Phi(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} \Phi(1+a-b,2-b;z) \)

with

\[
(b)_0 = 1, \quad (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)\cdots(b+k-1) \text{ for } k \geq 1.
\]

\( \Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \), if \( \text{Re}(z) > 0 \)

is the Gamma function. The confluent hypergeometric function of the second kind is defined for all values of \( a \) and \( b \), and it is bounded with respect to \( z \) (see Lebedev, 1972).
decrease, and it decreases when \( E_i \) increases. Moreover, the smooth-pasting condition requires that the option term be increasing in \( x \) for both models since 

\[
\forall x \geq 0, \forall v \geq 0, \quad \frac{dP_{2,m}(x;v)}{dx} - \frac{dP_{1,m}(x;v)}{dx} > 0.
\]

From Equation (18), \( P_{1,m}(x;v) \) for Model 2 is infinite if 

\[
v = \frac{r + \frac{k\alpha}{k(k-1)}}{2}, \quad 2 \leq k \leq m.
\]

Since these ratios are decreasing in \( k \), \( v \) has to be smaller than 

\[
\frac{r + \frac{m\alpha}{m(m-1)}}{2},
\]

which gives Equation (19). We see that the larger \( m \), the smaller the range of \( v \) for finite social damages from pollution. The possibility of unbounded expected social damage for a finite infinitesimal variance is one reason for studying Model 2 in parallel with Model 1. It illustrates the importance of knowing the process driving a stochastic variable. Knowing just the magnitude of its variations is obviously insufficient.

To find the option terms, we look for a solution to the homogeneous equation associated to Equation (6), which is well defined at \( X=0 \) and increasing in \( X \). Let us start with Model 1. The change of variables 

\[
Y = \frac{2\alpha X}{v}
\]

and \( W(Y) = V(X) \), leads to:

\[
\begin{align*}
Y \frac{d^2W}{dY^2} + (\frac{2E_i}{v} - Y) \frac{dW}{dY} - \frac{r}{\alpha} W = 0
\end{align*}
\]

This is Kummer’s Equation (see Lebedev, 1972). A general solution of this second order, ordinary differential equation, can be written:

\[
W(Y) = A_0 \Phi\left(\frac{r}{\alpha}, \frac{2E_i}{v}; Y\right) + B_0 Y^{-\frac{2E_i}{r}} \Phi\left(\frac{r}{\alpha}, \frac{2E_i}{v}; Y\right)
\]

If \( B_0 \) were non-zero, the second term on the right-hand side of Equation (24) would cause 

\[
\frac{\partial \tilde{\phi}(0;v)}{\partial x}
\]

to be infinite. We thus set \( B_0 \) to 0 and obtain Equation (17).
For Model 2, the change of variables
\[ Y = \frac{2E_1}{vX} \quad \text{and} \quad \left( \frac{E_1}{X} \right)^\beta W \left( \frac{2E_1}{vX} \right) = V(X) \]
leads again to Kummer’s equation, but this time we write its solution as follows:

\[
V(X;v) = A_0 \left( \frac{E_1}{X} \right)^\beta \Phi(\beta, \xi; 2E_1 \frac{2E_1}{vX}) + B_0 \left( \frac{E_1}{X} \right)^\beta \Psi(\beta, \xi; 2E_1 \frac{2E_1}{vX})
\]

Again, we want the option term to be well defined at \( X=0 \), and increasing in \( X \). Since
\[
X^{-\beta} \Phi(\beta, \xi, 2E_1 \frac{2E_1}{vX}) = \frac{\Gamma(\xi)}{\Gamma(\beta)} e^{\frac{2E_1}{vX} \xi \ln(x)} [1 + o(X)],
\]
the first term on the right-hand side of Equation (25) grows unbounded when \( X \to 0^+ \). From (4.1.12) in Slater (1960), \( \Psi(a,b,z) \approx z^a \) when \( z \to +\infty \), so the second term on the right-hand side is equivalent to
\[
B_0 \left( \frac{2E_1}{vX} \right)^{-\beta}
\]
when \( X \to 0^+ \). It can thus be defined by continuity at \( X=0 \). Setting \( A_0 \) to zero yields Equation (20).

Introducing Equations (16) and (17) into the continuity and smooth-pasting conditions gives the stopping rule for Model 1. The critical stock of pollutant, \( x^* \), is the smallest non-negative real which verifies:

\[
\Phi \left( \frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v} x^* \right) \left( \frac{r}{\alpha} + 1, \frac{2E_1}{v} + 1; \frac{2\alpha}{v} x^* \right) = \frac{P_{2,m}(x^*;v) - P_{1,m}(x^*;v) - K}{P_{2,m}(x^*;v) - P_{1,m}(x^*;v)}
\]

Likewise, with Equations (18) and (20), the stopping rule for Model 2 is:

\[
\frac{x^*}{\beta} \Psi \left( \frac{\beta, \xi; 2E_1}{vX} \right) - \Psi \left( \frac{\beta, \xi; 2E_1}{vX} \right) = \frac{P_{2,m}(x^*;v) - P_{1,m}(x^*;v) - K}{P_{2,m}(x^*;v) - P_{1,m}(x^*;v)}
\]

Both equations define implicitly \( x^* \) as a function of \( v \). They have the same interpretation as the deterministic stopping rule.
In general, it is not possible to find an explicit expression for \( x^* \) from the stopping rules, and it is quite difficult to examine how \( x^* \) changes with \( v \) because of the complexity of the derivative of the hypergeometric functions with respect to their second parameter. We thus examine how \( x^* \) changes for “large” and “small” values of \( v \). Considering first “large” values of \( v \), we have:

**PROPOSITION 1.** For “large enough” values of \( v \), \( x^*(v) \) decreases to zero. There is at most one non-negative value of \( v \), noted \( \bar{v} \), which corresponds to \( x^*=0 \). For Model 1, it verifies:

\[
(28) \quad \frac{c \, m!}{r + m\alpha} \frac{E_1 - E_2}{E_2} \prod_{i=0}^{m-1} \left( \frac{E_2}{E_2} \right) + l^* \cdot 0.5^* \left( \frac{\bar{v}}{r + l\alpha} \right) = K
\]

For Model 2, it verifies:

\[
(29) \quad \frac{E^{m-1} \frac{E_1 - E_2}{E_2}}{r} = \frac{K \prod_{i=1}^{m} (r + l\alpha - k(l-1)\bar{v}/2)}{cm!}
\]

If there is no such \( v \), pollutant emissions should be reduced to \( E_2 \) right away.

This result could have been anticipated because an increase in \( v \) augments expected social costs, as remarked above, but leaves \( K \) unchanged. Environmental damage ends up prevailing over the sunk costs of reducing the emissions of pollutant.

**Proof of PROPOSITION 1.** Let us start with Model 1. We obtain Equation (28) by setting \( x^* \) to zero in Equation (26) and simplifying. The left-hand side of Equation (28) is a polynomial in \( \bar{v} \) with non-negative coefficients, so it is strictly increasing in \( \bar{v} \). Since the right-hand side of Equation (28) is constant, it has at most one solution. Hence, there is a unique solution when \( K \geq \frac{c \, m!}{r + m\alpha} \frac{E_1 - E_2}{E_2} \prod_{i=0}^{m-1} \left( \frac{E_2}{E_2} \right) \). If \( K \) is smaller than this value, i.e. if the sunk investment needed for cutting emissions to \( E_2 \) is “small” relative to
expected social gains from reducing pollution, then pollutant emissions should be cut right away.

For Model 2, we proceed the same way. We start by setting \( x^* \) to zero in Equation (27). Its right-hand side at \( x^*=0 \), noted \( R(v) \), is:

\[
R(v) = \frac{1}{r} \frac{E_1^m - E_2^m}{E_1^{m-1} - E_2^{m-1}} \frac{K \prod_{j=1}^{m} (r + l \alpha - l(l - 1)v/2)}{c m! (E_1^{m-1} - E_2^{m-1})}
\]

From Lebedev (1972, p. 270), \( U(\alpha, \gamma, z) = z^{-\alpha} (1 - \frac{\alpha(1 + \alpha - \gamma)}{z} + \alpha(\frac{1}{z})) \) when \( z \to +\infty \), so the left-hand side of Equation (27) at \( x^*=0 \), noted \( L(v) \), simplifies to:

\[
L(v) = \frac{E_1}{r}
\]

Equating \( L(v) \) to \( R(v) \) and rearranging terms gives Equation (29).

When \( m=2 \), we can derive an explicit expression for \( \tilde{v} \), valid for \( x_0^* > 0 \):

\[
\tilde{v} = \begin{cases} 
2(r + \alpha)x_0^*, & \text{for Model 1} \\
\frac{(r + \alpha)(r + 2\alpha)x_0^*}{(r + \alpha)x_0^* + E_2}, & \text{for Model 2}
\end{cases}
\]

Let us now investigate how \( x^* \) changes when \( v \) goes to zero. We have:

**PROPOSITION 2.** Let \( \tilde{x}_0^* = \lim_{v \to 0^+} x^*(v) \). When \( x_0^* < x_{1c} \), \( \tilde{x}_0^* = x_0^* \), but when \( x_0^* > x_{1c} \), \( \tilde{x}_0^* \neq x_0^* \). Moreover, \( \tilde{x}_0^* \) is the smallest non-negative value of \( x \) that verifies:

\[
[p_{2,m}(x;0) - p_{1,m}(x;0) - K] = 0.
\]

The limit when \( v \) goes to zero of the critical value of the stock of pollutant, \( x^*(v) \), may thus be different from \( x_0^* \), the deterministic critical value. In addition, for small values of \( v \),
$x^*(v)$ may be larger or smaller than $\tilde{x}_0^*$, depending on the cost of reducing pollutant emissions relative to the gain of reducing expected environmental damage.

Proof of PROPOSITION 2. We use the same approach for both models. We outline only the main steps of the proof. More details are provided in Appendix B. First, we expand each side of the stopping rules in terms of $v$, and then substitute in the first order expansion of $x^*(v)$. Designating the left and right-hand side of the stopping rules by $LHS(x^*;v)$ and $RHS(x^*;v)$ respectively, we find for Model 1:

\[
LHS(x^*;v) = \begin{cases} 
    \frac{E_1 - \alpha \tilde{x}_0^*}{r} + \left( \frac{\tilde{x}_0^*(r + \alpha)}{2} - \frac{\alpha}{r} \frac{dx^*(0)}{dv} \right) r + o(v), & \text{if } x^* < x_{1c} \\
    \frac{\tilde{x}_0^*}{2(\alpha \tilde{x}_0^* - E_1)} + o(v), & \text{if } x^* > x_{1c}
\end{cases}
\]

For Model 2, we get:

\[
LHS(x^*;v) = \begin{cases} 
    \frac{E_1 - \alpha \tilde{x}_0^*}{r} + \left( \frac{E_1^2}{2\alpha^2 r (E_1 - \alpha \tilde{x}_0^*)} - \frac{\alpha}{r} \frac{dx^*(0)}{dv} \right) r + o(v), & \text{if } x^* < x_{1c} \\
    \frac{\tilde{x}_0^*}{2(\alpha \tilde{x}_0^* - E_1)} + o(v), & \text{if } x^* > x_{1c}
\end{cases}
\]

RHS($x^*;v$) has the same form for both models:

\[
RHS(x^*;v) = \frac{\tilde{p}_m}{\tilde{p}} + \left[ \frac{\tilde{p}_m^2 - \tilde{p}_m \tilde{p}^*}{\tilde{p}^2_m} \frac{dx^*(0)}{dv} + \frac{\tilde{p}_m \tilde{q}_m - \tilde{p}_m \tilde{q}_m^*}{\tilde{p}^2_m} \right] + o(v)
\]

$\tilde{p}_m$ is an abbreviation for $P_m(\tilde{x}_0^*) \cdots P_{2m}(\tilde{x}_0^*;0) - P_m(\tilde{x}_0^*;0) - K$, which represents the net benefit of reducing emissions at $\tilde{x}_0^*$; $\tilde{p}_m^*$ and $\tilde{p}_m^"$ are respectively the first and second derivatives, with respect to $x$, of $P_m(x)$ at $\tilde{x}_0^*$; $\tilde{q}_m^*$ and $\tilde{q}_m^"$ are respectively the first and second derivatives, with respect to $x$, of $Q_m(x)$ at $\tilde{x}_0^*$ where:

\[
Q_m(x) = e^{\sum_{k=0}^{m} (m-k)(m+k-1) \frac{m!(E_1^{m-k-1} - E_2^{m-k-1})}{k! \prod_{l=k}^{m} (r + l\alpha)} x^k}
\]

for Model 1, and
for Model 2; finally, $\tilde{Q}_m$ is the derivative of $Q_m(x)$ at $\tilde{x}_0^*$.

Equating the constant terms in LHS($x^*;v$) and RHS($x^*;v$) proves the first part of this proposition. When the stock of pollutant is above $x_{1c}$, the deterministic stopping rule (Equation (14)) is thus inadequate for telling us when to take action to reduce pollutant emissions, even when uncertainty is low. This is because there is a qualitative difference between the deterministic case, where $x_{1c}$ is a “barrier” for the stock of pollutant, and the stochastic case, where $X$ can take any non-negative value, even when $v$ is small. We also note that while $x^*(v)$ may be discontinuous at $v=0$, the option value function is continuous in $v$. Recall indeed that the deterministic option value is zero when $x_0^*>x_{1c}$.

In the stochastic case, Equation (33) forces the option term constants, $A_0$ and $B_0$, to be zero at $v=0$, since the confluent hypergeometric functions with positive arguments are strictly positive.

When $m=2$, $\tilde{x}_0^*$ is related to $x_0^*$ by:

$$\tilde{x}_0^* = \left( \frac{r + \alpha}{r} \right) x_0^* - E_1$$

Equating the first order terms in LHS($x^*;v$) and RHS($x^*;v$) leads to fairly complex expressions for $\frac{dx^*(0)}{dv}$. These expressions (one for each stochastic case) can be somewhat simplified for particular values of $\tilde{x}_0^*$. For Model 1, this leads to:
\[ \frac{dx^*(0)}{dv} = \begin{cases} \frac{E_1}{2 \alpha |\epsilon|} > 0, & \text{when } \tilde{x}_0^* = \frac{E_1 \pm \epsilon}{\alpha}, \text{ with } |\epsilon| \text{ small} \\ \frac{r - (m - 1) \alpha}{2 \alpha [r + (m - 2) \alpha]}, & \text{when } \tilde{x}_0^* \gg \frac{E_1}{\alpha} \end{cases} \]

For Model 2, we obtain:

\[ \frac{dx^*(0)}{dv} = \begin{cases} \frac{E_1}{2 \alpha |\epsilon|} > 0, & \text{when } \tilde{x}_0^* = \frac{E_1 \pm \epsilon}{\alpha}, \text{ with } |\epsilon| \text{ small} \\ \frac{x_0^* r + \alpha}{2 \alpha (r + m \alpha)}, & \text{when } \tilde{x}_0^* \gg \frac{E_1}{\alpha} \end{cases} \]

When K is low compared to expected reduction in social pollution costs (i.e. when \( \tilde{x}_0^* \) is “close to zero”), \( \frac{dx^*(0)}{dv} \) is negative for Model 1 so \( x^*(v) \) decreases as \( v \) increases.

For Model 2, the infinitesimal variance is smaller in this case, so expected social pollution costs may grow more slowly with \( v \); \( \frac{dx^*(0)}{dv} \) can thus be positive if \( r/\alpha \) is large enough (the future is not that important or the pollutant decays quickly). When K is higher and \( \tilde{x}_0^* \) is close to \( x_{1c} \), \( \frac{dx^*(0)}{dv} \) tends to \( +\infty \) for both models, because pollution control costs dominates. Finally, for still larger values of K, \( x^*(0) \) is greater than \( x_{1c} \). For Model 1, \( \frac{dx^*(0)}{dv} \) is negative if the social discount rate \( r < (m - 1) \alpha \), and it is positive otherwise. For Model 2, however, \( \frac{dx^*(0)}{dv} \) is always positive. These fairly complex results illustrate the richness of a continuous time formulation.
A numerical illustration is presented in Tables I and II and in Figures I and II. Calculations were performed with Mathcad 7.0. Tables I and II focus on the effect of small but increasing volatility of the stock of pollutant on $x^*(v)$ and the option value at $x^*(v)$ normalized by $K$ (noted $\varphi^*/K$). As shown in Proposition 2, $x^*(v)$ increases or decreases with $v$ relative to $\bar{x}_0^*$. We also see that changes in $x^*(v)$ and in the corresponding normalized option value can be quite steep as $v$ increases from 0, especially for Model 2 (look at the last column of Table II). For the values of the parameters explored, we find that the option value at $x^*(v)$ increases with $v$. Moreover, we see that the option value can be of the same order of magnitude as $K$ (see for example the cases $r=0.02$ and $\alpha=0.03$ for Model 1 or $r=0.03$ and $\alpha=0.03$ for Model 2). When $x^*(0)>x_{1c}$, the option value is zero when $v=0$, as in the deterministic case, but it can increase sharply with $v$, especially for Model 2. Ignoring option value, as is commonly done in a static cost benefit analysis, can thus lead to very sub-optimal decisions. In addition, we note again that while $x^*(v)$ can be discontinuous at $v=0$ (when $x_0^*>x_{1c}$), the option value is continuous in $v$ at 0.

Figures I and II give a global view of the changes of $x^*(v)$ with $v$ for different values of $r$ when the damage function is quadratic in $X$ ($m=2$). These results were obtained for $E_1=1, E_2=0.7$ (i.e. a 30% cut in pollutant emissions), $K=6000$, and $c=1.0$. Although we use slightly different values of $\alpha$ between the two figures, we can see that $x^*(v)$ is much more sensitive to $v$ for Model 2 than for Model 1. This could be seen from
the expressions of the expected social damage functions (Equations (16) and (18)). As
proved in Proposition 1, \( x^*(v) \) goes to zero for large enough values of \( v \) for both models.

Finally, these calculations illustrate the great sensitivity of \( x^*(v) \) to \( r \) and \( \alpha \).

VI. CONCLUSIONS

In this paper, we use real options to analyze the tension between environmental
irreversibility and sunk costs to reduce the emissions of a pollutant, when the stock of
this pollutant varies randomly due to environmental factors. Starting with a deterministic
model, we get a benchmark for evaluating the impact of uncertainty, and we illustrate the
concept of real option. For us, it is the value of the flexibility to choose the timing and
intensity of reducing the emissions of a pollutant. We then consider two stochastic
models. One gives finite expected social damage for all values of the volatility of the stock
of pollutant, but not the other. Starting from no uncertainty, we find that increasing the
level of uncertainty may either advance or delay a reduction in pollutant emissions,
depending on the cost of cutting emissions relative to the gain in reducing expected social
damage from pollution. When uncertainty is “high enough”, we find that pollutant
emissions should be reduced right away, because increasing levels of uncertainty lead to
increasing expected social damage while the cost of cutting emissions is fixed. These
results hold for both models explored.

These results complement what is already known in the environmental and
resource literature, where a standard result is that uncertainty in the presence of
environmental irreversibilities will delay the development of a natural resource (Arrow
and Fisher, 1974). More recently, Pindyck (in Dixit and Pindyck, 1994) explores a model
similar to ours with a social damage function quadratic in the stock of pollutant. He shows that, when the coefficient of valuation of social damage from pollution, noted c, follows a geometric Brownian motion, the critical value of the stock of pollutant at which emissions should be cut, increases with the volatility of c. Taking into account environmental and economic irreversibilities, we obtain a more complex behavior with only one stochastic variable.

An important finding of this paper is that it may not be adequate to use a deterministic model to study a stochastic problem, even when uncertainty is “small.” In any case, a comparison between our two stochastic models shows that a “small” level of uncertainty may have benign effects in some cases (Model 1) but may lead to very large damages in other cases (Model 2). Although our models are too simple to represent any real situation, we should interpret with caution the findings of purely deterministic models, especially for such important and inherently stochastic environmental problems as global warming.

Finally, this paper illustrates that the theory of real options is a very useful tool for analyzing environmental policy, because irreversibility and uncertainty are dominant features of many environmental problems. While the real option approach contains Weisbrod’s initial intuition on option value, it does not suffer from some of the conceptual problems that plague the definition of option value in the environmental economics literature. The complexity of multivariate stochastic problems and the lack of data to estimate more complex models may, however, limit for now its widespread use.

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REFERENCES


**APPENDIX A**

In this appendix, we derive the moment generating functions for the two diffusions considered in this paper. We first summarize the method used, and we then provide the main intermediate results for each model. Let $X(t)$ be a diffusion process which verifies:

\begin{equation}
\frac{dX}{dt} = a(X)dt + b(X,t)dz
\end{equation}

where $dz$ is an increment of a standard Wiener process. The moment generating function of $X(t)$, noted $M(\theta,t)$, is defined by:

\begin{equation}
M(\theta,t) = E(e^{-\theta X}) = \int_{-\infty}^{\infty} \phi(x_0,t_0;x,t)e^{-\theta x}dx
\end{equation}

$\phi(x_0,t_0;x,t)$ is the probability density function for $x$ at $t$, given $x(t_0)=x_0$. Then:

\begin{equation}
\frac{\partial M}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial t} e^{-\theta x}dx
\end{equation}

To derive $M(\theta,t)$, we insert the left-hand side of the Kolmogorov forward equation:

\begin{equation}
\frac{1}{2} \frac{\partial^2}{\partial x^2} (b^2(x,t)\phi(x_0,t_0;x,t)) - \frac{\partial}{\partial x} (a(x,t)\phi(x_0,t_0;x,t)) = \frac{\partial}{\partial t} \phi(x_0,t_0;x,t)
\end{equation}

into (A3), integrate by parts, and solve the resulting partial differential subject to the boundary conditions:
(A5) \[ M(0,t) = 1, \quad \frac{\partial M(0,0)}{\partial \theta} = -x_0, \quad \frac{\partial^2 M(0,0)}{\partial \theta^2} = x_0 \]

- **Model 1**

For the process:

(A6) \[ dX = (-\alpha X + E)dt + \sqrt{v}Xdz, \]

the Kolmogorov forward equation is:

(A7) \[ \frac{\partial \phi}{\partial t} = \frac{vx}{2} \frac{\partial^2 \phi}{\partial x^2} + (v + \alpha x - E) \frac{\partial \phi}{\partial x} + \alpha \phi \]

After substituting (A7) into (A3), we integrate to obtain:

(A8) \[ \frac{\partial M}{\partial t} = -\theta (\alpha + \frac{v}{2} \theta) \frac{\partial M}{\partial \theta} - E\theta M \]

Solving this partial differential equation subject to the boundary conditions (A5), we find:

(A9) \[ M(\theta, t) = \left(1 + \frac{v\theta}{2\alpha}\right)^{\frac{2E}{\nu}} \left[1 + C_1 \frac{2\theta e^{-\alpha t}}{2\alpha + \theta v} + C_2 \left(\frac{2\theta e^{-\alpha t}}{2\alpha + \theta v}\right)^2\right] \]

with \( C_1 = E - \alpha x_0 \), \( C_2 = \frac{1}{2} \left(\left(E - \alpha x_0 + \frac{v}{2}\right)^2 - \frac{v}{2} \left(E + \frac{v}{2}\right)\right) \)

- **Model 2**

For the process:

(A10) \[ dX = (-\alpha X + E)dt + \sqrt{v}Xdz, \]

the Kolmogorov forward equation is:

(A11) \[ \frac{\partial \phi}{\partial t} = \frac{vx}{2} \frac{\partial^2 \phi}{\partial x^2} + (2vx + \alpha x - E) \frac{\partial \phi}{\partial x} + (v + \alpha) \phi \]

After substituting (A11) into (A3), we integrate to obtain:

(A12) \[ \frac{\partial M}{\partial t} = \frac{v}{2} \theta^2 \frac{\partial^2 M}{\partial \theta^2} - \theta\alpha \frac{\partial M}{\partial \theta} - E\theta M \]

The solution to this partial differential equation with boundary conditions (A5) is:
We start by deriving the approximation of the left-hand side of each stopping rule.

- Model 1

When \( v \to 0 \), we have the formal convergence:

\[
\Phi \left( \frac{r}{\alpha}; \frac{2E_i}{v}; \frac{2\alpha}{v} x^*(v) \right) \to S(\tilde{x}_0^*) \equiv \sum_{n=0}^{+\infty} \left( \frac{r}{\alpha} \right) \frac{1}{n!} \left( \frac{\alpha \tilde{x}_0^*}{E_i} \right)^n
\]

The series on the right side of Equation (A14) converges towards \( 1 - \frac{\alpha \tilde{x}_0^*}{E_i} \) provided

\[
\frac{\alpha \tilde{x}_0^*}{E_i} < 1 \), or \( \tilde{x}_0^* < \frac{E_i}{\alpha} \equiv x_{1c} \), where \( \tilde{x}_0^* = \lim_{v \to 0} x^*(v) \). When this condition does not hold, \( S(\tilde{x}_0^*) = +\infty \). We thus need to distinguish between two separate cases.

When \( \tilde{x}_0^* < x_{1c} \), we use the following contiguity relationship (from Slater):

\[
b(b+1)\Phi(a,b;y) + (b+1)(y-b)\Phi(a+1,b+1;y) - (a+1)y\Phi(a+2,b+2;y) = 0
\]

to write:

\[
\frac{\Phi(a,b,bz)}{\Phi(a+1,b+1,bz)} = 1 - z + \frac{(a+1)z}{(b+1)\Phi(a+1,b+1;bz)} \frac{\Phi(a+1,b+1;bz)}{\Phi(a+2,b+2;bz)}
\]

We substitute \( a = \frac{r}{\alpha}, b = \frac{2E}{v}, z = \frac{\alpha x}{E} \) in the above and simplify to find the first part of Equation (34).
When \( x_0^* > x_{1c} \), we use (9.12.8) in Lebedev to write:

\[
(A17) \quad \Phi(a,b,by) = \frac{\Gamma(b)}{\Gamma(a)} e^{by} (by)^{a-b} \left\{ (1 - \frac{1}{y})^{a-1} + o(1) \right\}
\]

We substitute \( a = \frac{r}{\alpha} \), \( b = \frac{2E}{v} \), \( z = \frac{\alpha x}{E} \) in the above and simplify to find

\[
(A18) \quad \Phi\left(\frac{r}{\alpha}, \frac{2E}{v}, \frac{2\alpha x}{v} x^*\right) = \frac{\Gamma\left(\frac{2E}{v}\right)}{\Gamma\left(\frac{r}{\alpha}\right)} e^{\frac{2\alpha x}{v} x^*} \left\{ \left(1 - \frac{E}{\alpha x^*}\right)^{\frac{r}{\alpha}} + o(1) \right\}
\]

We repeat this exercise for the derivative with respect to \( x \) of \( \Phi(a,b,x) \). Combining the result with (A18), we find the second part of Equation (34).

- **Model 2**

When \( v \to 0 \), we use (9.12.3) in Lebedev (Chap. 9, p. 270) to write:

\[
(A19) \quad \Psi(\beta, \xi; \frac{2E}{vx}) = \left(\frac{2E}{vx}\right)^{-\frac{r}{\alpha}} \left\{ \left[1 - \frac{\alpha x}{E}\right]^{\frac{r}{\alpha}} + o(1) \right\}
\]

Likewise:

\[
(A20) \quad \Psi(\beta + 1, \xi + 1; \frac{2E}{vx}) = \left(\frac{2E}{vx}\right)^{-\frac{r}{\alpha}} \left\{ \left[1 - \frac{\alpha x}{E}\right]^{-\frac{r}{\alpha}} + o(1) \right\}
\]

To obtain a first order approximation in \( v \) of the ratio of \( \Psi(a,b,y) \) by \( y \Psi(a+1,b+1,y) \), we use the following contiguity relationship derived from Slater:

\[
(A21) \quad \Psi(a,b;y) + (b - y)\Psi(a+1,b+1;y) - y(a+1)\Psi(a+2,b+2;y) = 0
\]

Hence:

\[
(A22) \quad \frac{\Psi(a,b,bz)}{bz\Psi(a+1,b+1,bz)} = 1 - \frac{1}{z} + \frac{a + 1}{\Psi(a+1,b+1,bz)} \cdot \frac{\Psi(a+1,b+1,bz)}{\Psi(a+2,b+2,bz)}
\]
We substitute for $\beta$, $\xi$, and $x^*(v)$, introduce a first order expansion of $x^*(v)$ in terms of $v$, and use the resulting expression in the left-hand side of Equation (27) to obtain the first part of Equation (35).

When $\tilde{x}_0^* > x_{1c}$, we start by looking for the limit of

$$\frac{\Psi(\beta, \xi; \frac{2E}{v_x})}{\Psi(\beta + 1, \xi + 1; \frac{2E}{v_x})}$$

as $v \to 0^+$. To simplify the the equivalent of $\Psi$ when $v \to 0^+$, we consider the sequence $(v_n)$ such that

$$\frac{2\alpha}{v_n} + 2 + \frac{2r}{\alpha} = 2n + \frac{1}{2}.$$ We find:

$$\lim_{v \to 0^+} \frac{\Psi(\beta, \xi; \frac{2E}{v_x})}{\Psi(\beta + 1, \xi + 1; \frac{2E}{v_x})} = \frac{r}{\alpha \alpha \alpha - E} \frac{E}{\alpha \alpha \alpha - E}$$

We substitute this result into the left-hand side of Equation (27) to obtain the second part of Equation (35).

Finding an approximation to the right-hand side of each stopping rule requires only simple, albeit tedious, algebra. The key relationship for Models 1 and 2 are respectively:

(A24) \[ \prod_{j=k}^{m-1} (E + j \frac{v}{2}) = E^{m-k} \left(1 + \frac{v}{2E} (m - k)(m + k - 1) \right) + o(v) \]

(A25) \[ \prod_{j=k}^{m} (r + j\alpha - \frac{v}{2} j(j - 1)) = \frac{\prod_{j=k}^{m} (r + j\alpha)}{1 + \frac{v}{2} \sum_{j=k}^{m} \frac{j(j - 1)}{r + j\alpha}} + o(v) \]

Introducing these relationships into the right-hand sides of the stopping rules leads to Equation (36).
Equating the approximate left-hand side to the approximate right-hand side of each stopping rule gives a complex expression for the derivative with respect to \( v \) of \( x^*(v) \) at 0+. We obtain respectively for Model 1 and for Model 2:

(A26) \[
\frac{dx^*(0)}{dv} = \begin{cases} 
x^* \left( r + \alpha \right) \\
\frac{2r(E - \alpha x^*_m)}{2r(E - \alpha x^*_0)} - \frac{\tilde{P}_m \tilde{Q}_m - \tilde{P}_m \tilde{Q}_m}{\tilde{P}_m^2}, \text{ when } \tilde{x}_0^* < x_{1c} \\
\frac{\alpha}{r} + \frac{\tilde{P}_m^2 - \tilde{P}_m \tilde{P}_m}{\tilde{P}_m^2} \\
\frac{\tilde{x}_0^*}{2(\alpha \tilde{x}_0^* - E)} - \frac{\tilde{Q}_m^2}{\tilde{P}_m}, \text{ when } \tilde{x}_0^* > x_{1c}
\end{cases}
\]

(A27) \[
\frac{dx^*(0)}{dv} = \begin{cases} 
E^2 \left( r + \alpha \right) \\
\frac{2\alpha^2 r(E - \alpha x^*_0)}{2\alpha^2 r(E - \alpha x^*_m)} - \frac{\tilde{P}_m \tilde{Q}_m - \tilde{P}_m \tilde{Q}_m}{\tilde{P}_m^2}, \text{ when } \tilde{x}_0^* < x_{1c} \\
\frac{\alpha}{r} + \frac{\tilde{P}_m^2 - \tilde{P}_m \tilde{P}_m}{\tilde{P}_m^2} \\
\frac{\tilde{x}_0^*}{2(\alpha \tilde{x}_0^* - E)} - \frac{\tilde{Q}_m^2}{\tilde{P}_m}, \text{ when } \tilde{x}_0^* > x_{1c}
\end{cases}
\]

The terms \( \tilde{P}_m, \tilde{P}_m', \tilde{P}_m'', \tilde{Q}_m \) and \( \tilde{Q}_m' \) are defined in the text for each model. \( \frac{dx^*(0)}{dv} \) can be simplified when \( \tilde{x}_0^* > x_{1c} \) because \( \tilde{P}_m = 0 \) in that case. Moreover, this derivative is not defined at \( x_{1c} \) because it is a singular point for the deterministic model: either the stock of pollutant stays at that value forever, or it never reaches it. Finally, when we consider special values of \( \tilde{x}_0^* \) to look at the sign of \( \frac{dx^*(0)}{dv} \), we derive Equations (40) and (41).
Table I: $X^*(v)$ for Model 1.

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<th>$\sqrt{v}$</th>
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<th>$\varphi^*/K$</th>
<th>$x^*$</th>
<th>$\varphi^*/K$</th>
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<td>0.0000</td>
<td>8.33</td>
<td>55.6%</td>
<td>23.00</td>
<td>3.5%</td>
<td>53.50</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.0025</td>
<td>8.33</td>
<td>55.7%</td>
<td>23.27</td>
<td>3.9%</td>
<td>53.58</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td>0.0050</td>
<td>8.32</td>
<td>55.7%</td>
<td>23.48</td>
<td>4.2%</td>
<td>53.58</td>
<td>0.1%</td>
<td></td>
</tr>
</tbody>
</table>

Table II: $X^*(v)$ for Model 2.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sqrt{v}$</th>
<th>$x^*$</th>
<th>$\varphi^*/K$</th>
<th>$x^*$</th>
<th>$\varphi^*/K$</th>
<th>$x^*$</th>
<th>$\varphi^*/K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>0.0000</td>
<td>15.33</td>
<td>33.3%</td>
<td>30.00</td>
<td>3.6%</td>
<td>54.00</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.0025</td>
<td>15.04</td>
<td>36.6%</td>
<td>32.49</td>
<td>9.9%</td>
<td>56.36</td>
<td>5.1%</td>
<td></td>
</tr>
<tr>
<td>0.0050</td>
<td>14.60</td>
<td>39.7%</td>
<td>33.04</td>
<td>13.6%</td>
<td>57.35</td>
<td>8.8%</td>
<td></td>
</tr>
<tr>
<td>0.04</td>
<td>0.0000</td>
<td>23.00</td>
<td>3.5%</td>
<td>53.50</td>
<td>0.0%</td>
<td>83.00</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.0025</td>
<td>24.94</td>
<td>8.5%</td>
<td>54.33</td>
<td>3.0%</td>
<td>84.15</td>
<td>3.0%</td>
<td></td>
</tr>
<tr>
<td>0.0050</td>
<td>25.37</td>
<td>11.6%</td>
<td>54.71</td>
<td>5.7%</td>
<td>84.82</td>
<td>5.6%</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.0000</td>
<td>47.33</td>
<td>0.0%</td>
<td>83.50</td>
<td>0.0%</td>
<td>116.00</td>
<td>0.0%</td>
</tr>
<tr>
<td>0.0025</td>
<td>47.28</td>
<td>1.9%</td>
<td>83.86</td>
<td>2.1%</td>
<td>116.83</td>
<td>2.3%</td>
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</tr>
<tr>
<td>0.0050</td>
<td>47.03</td>
<td>3.7%</td>
<td>83.97</td>
<td>4.1%</td>
<td>117.35</td>
<td>4.4%</td>
<td></td>
</tr>
</tbody>
</table>

Note: These results were calculated with $E1=1$, $E2=0.7$, $K=6000$, $c=1.0$, and $m=2$.  

31
Figure I: $X^*$ vs. $v$ for Model 1 with $E_1=1$, $E_2=0.7$, $K/c=6000$, and $\alpha=0.03$

Figure II: $X^*$ vs. $v$ for Model 2 with $E_1=1$, $E_2=0.7$, $K/c=6000$, and $\alpha=0.025$